E.Xolmurodov, A.I. Yusupov
T.A. Aliqulov

OLIY MATEMATIKA

$$\int_a^b f(x)dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(z_k)\Delta x_k$$

$$f(x+\Delta x, y+\Delta y) \approx f(x,y) + \frac{\partial f(x,y)}{\partial x}\Delta x + \frac{\partial f(x,y)}{\partial y}\Delta y$$

2

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

E.XOLMURODOV, A. I. YUSUPOV, T. A. ALIQULOV

OLIY MATEMATIKA

II-QISM

O'zbekiston Respublikasi Oliy va o'rta maxsus ta'lim vazirligi tomonidan o'quv qo'llanma sifatida tavsiya etilgan

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Mazkur o'quv qo'llanma "300000 – ishlab chiqarish texnologiyalari sohasi" bilim sohasi ta'lim yo'nalishlari uchun tasdiqlangan "Oliy matematika" fanining namunaviy fan dasturi asosida yozilgan bo'lib, unda Oliy matematikaning kompleks sonlar, aniqmas va aniq integrallar, xosmas integral, aniq integralni geometriya va mexanikaga tadbiqlari, bir necha o'zgaruvchining funksiyasi bo'limlari bayon etilgan.

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SO'Z BOSHI

Ushbu o'quv qo'llanma «Ishlab chiqarish texnologiyalari sohasi» bilim sohasi ta'lim yo'nalishlari uchun tasdiqlangan «Oliy matematika» fanining namunaviy o'quv dasturi asosida tayyorlangan uch qismdan iborat o'quv qo'llanmaning ikkinchi qismidir.

O'quv qo'llanmaning ikkinchi qismi birinchi qismining uzviy davomi bo'lganligi sababli yirik mavzularni, formulalarni va chizmalarni raqamlanish ketma-ketligi saqlab qolingan. Ikkinchi qism o'quv qo'llanmaga oliy matematikaning kompleks sonlar, kompleks o'zgaruvchining funksiyasi, ko'phadlar, aniqmas va aniq integrallar, xosmas integrallar, aniq integralning geometriya va mexanikaga tadbiqlari, ko'p o'zgaruvchining funksiyasi bo'limlari kiritilgan.

O'quv qo'llanma mualliflarining uzoq yillar davomida «Oliy matematika» fanidan talabalarga o'qigan ma'ruzalari asosida yozilgan bo'lib, uni tayyorlashda talabalarni fanni chuqur o'zlashtirishiga qulaylik tug'diruvchi holatlariga tajribadan kelib chiqib yondashilgan.

O'quv qo'llanmada o'quvchining mavzuni o'zlashtirishi oson bo'lishi uchun sodda amallarni bajarilishigacha batafsil ko'rsatilgan. Mavzuga oid ko'plab mashqlar yechib ko'rsatilgan. Har bir mavzuning oxirida mustaqil yechish uchun mashqlar, o'z-o'zini tekshirish savollari berilgan bo'lib, ular o'quvchining olgan bilimlarini yanada mustahkamlashga yordam beradi.

O'quv qo'llanmadan boshqa bilim sohalariga tegishli bakalavr ta'lim yo'nalishlari talabalari ham foydalanishlari mumkin.

Ushbu o'quv qo'llanma «Oliy matematika» fanidan «Ishlab chiqarish texnologiyalari sohasi» bilim sohasiga tegishli ta'lim yo'nalishlarida tahsil olayotgan talabalar uchun lotin yozuviga asoslangan o'zbek alifbosi va imlosida tayyorlangan dastlabki adabiyotlardan bo'lganligi uchun kamchiliklardan holi bo'lmasligi mumkin. O'quv qo'llanma haqidagi fikr-mulohazalaringizni Qarshi Muhandislik-iqtisodiyot instituti «Oliy matematika» kafedrasiga yuborishingizni so'raymiz.

28. KOMPLEKS SONLAR VA ULARNING GEOMETRIK TASVIRI HAMDA TRIGONOMETRIK SHAKLI

28.1. Kompleks son tushunchasining kiritilish sababi

Ma'lumki sonlar orasida eng qadimgisi natural son bo'lib u narsa va buyumlarni sanash zaruriyati tufayli paydo bo'lgan.

Shuningdek, butun, ratsional, irratsional, haqiqiy son kabi tushunchalar ham zaruriyat tufayli kiritilgandir.

Shunday masalalar ham uchraydiki ularni haqiqiy sonlar sohasidan chiqmasdan yechib bo'lmaydi. Ana shunday masalalardan birini qaraymiz.

Umumiy hadi

$$S_n = 1 - x^2 + x^4 - x^6 + \ldots + (-1)^n x^{2n} \qquad (\alpha)$$

ko'rinishga ega bo'lgan $\{S_n\}$ ketma-ketlikni qaraymiz, bunda x haqiqiy erkli o'zgaruvchi.

S_n ni soddaroq ko'rinishda yozish maqsadida (α) tenglikni har ikkala qismini x^2 ga ko'paytiramiz. U holda

$$x^2 S_n = x^2 - x^4 + x^6 - \ldots + (-1)^{n-1} x^{2n} + (-1)^n x^{2(n+1)} \qquad (\beta)$$

tenglikka ega bo'lamiz.

(α) va (β) tengliklarni hadma-had qo'shsak $S_n + x^2 S_n = 1 + (-1)^n x^{2(n+1)}$ yoki $(1 + x^2)S_n = 1 + (-1)^n x^{2(n+1)}$ kelib chiqadi. Bu tenglikni har ikkala qismini $1 + x^2$ ga bo'lib

$$S_n = \frac{1 + (-1)^n x^{2(n+1)}}{1 + x^2}$$

tenglikni hosil qilamiz. Endi $\{S_n\}$ ketma-ketlikni $n \to \infty$ dagi limitini topishga kirishamiz. Oxirgi tenglikda $n \to \infty$ da limitga o'tsak

$$\lim_{n\to\infty} S_n = \frac{1}{1+x^2}\lim_{n\to\infty}[1+(-1)^n x^{2(n+1)}] \qquad (\gamma)$$

bo'ladi, chunki $\frac{1}{1+x^2}$ ifoda n ga bog'liq bo'lmaganligi sababli uni o'zgarmas son sifatida limit ishorasidan chiqarildi.

$|x|<1$ bo'lganda $\lim_{n\to\infty} x^n = 0$ va $|x|>1$ bo'lganda $\lim_{n\to\infty} x^n = \infty$ ekanligini hisobga olib (γ) tenglikdan $|x|<1$ bo'lganda

$$\lim_{n\to\infty} S_n = \frac{1}{1+x^2}$$

tenglikka va $|x|>1$ bo'lganda

$$\lim_{n\to\infty} S_n = \infty$$

ga ega bo'lamiz.

$x=\pm 1$ bo'lganda $\{S_n\}$ ketma-ketlik

$$0,1,0,1,0,1,0,...$$

ko'rinishdagi limitga ega bo'lmagan ketma-ketlikka aylanadi.

Shunday qilib $\{S_n\}$ ketma-ketlik uchun quyidagi natijaga ega bo'ldik.

$|x|<1$ bo'lganda $\{S_n\}$ ketma-ketlik yaqinlashuvchi va $\frac{1}{1+x^2}$ limitga ega, ya'ni

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + ... \qquad (\delta)$$

tenglik o'rinli.

$|x| \geq 1$ bo'lganda $\{S_n\}$ ketma-ketlik uzoqlashuvchi. Bu holda $\frac{1}{1+x^2}$ funksiyani (δ) tenglikning o'ng tomonidagi yig'indi ko'rinishda tasvirlab bo'lmaydi.

Nima uchun $\frac{1}{1+x^2}$ funksiyani $|x|<1$ bo'lganda (δ) ko'rinishida tasvirlash mumkinu uni $|x|\geq 1$ bo'lganda bu ko'rinishida tasvirlash mumkin emas degan savolga x haqiqiy son bo'lganda javob berib bo'lmaydi.

Qo'yilgan savolga to'liq javobni kiritiladigan kompleks son tushunchasidan foydalanib olish mumkin.

Bu masalaga keyinroq yana qaytamiz.

28.2. Asosiy ta'riflar

Haqiqiy sonlar bilan ish ko'rilganda istalgan noldan farqli haqiqiy sonni kvadrati musbat son bo'lishini ko'rdik. Endi kvadrati manfiy son bo'lgan son tushunchasini kiritamiz. Bunday sonlar tabiiyki haqiqiy son bo'lmaydi.

1-ta'rif. Kvadrati -1 ga teng ifoda **mavhum birlik** deb ataladi va u i orqali belgilanadi.

Shunday qilib, $i^2=-1$ yoki $i=\sqrt{-1}$.

Mavhum birlikning ta'rifidan $i^3=i^2\cdot i=-1\cdot i=-i$, $i^4=i^2\cdot i^2=(-1)(-1)=1$, $i^5=-i$ va hokazo umuman k butun son uchun $i^{4k}=1$, $i^{4k+1}=i$, $i^{4k+2}=-1$, $i^{4k+3}=-i$ ekannligi kelib chiqadi. Shuningdek, $\frac{1}{i}=\frac{i}{i^2}=-i$.

2-ta'rif. $z=a+bi$

ko'rinishdagi ifodaga kompleks son deb aytiladi, bunda a va b haqiqiy sonlar.

a va b ni mos ravishda z kompleks sonning haqiqiy va mavhum qismlari deyiladi va $Rez=a$, $Jmz=b$ kabi belgilanadi.

Xususiy holda, agar $a=0$ bo'lsa u holda $z=0+ib=bi$ bo'lib u **sof mavhum** son deyiladi. Agar $b=0$ bo'lsa $z=a+i0=a$ haqiqiy son hosil bo'ladi. Demak, haqiqiy va sof mavhum sonlar kompleks sonning xususiy holi ekan.

3-ta'rif. Ikkita $z_1=a_1+ib_1$ va $z_2=a_2+ib_2$ kompleks sonlar $a_1=a_2$ $b_1=b_2$ bo'lgandagina teng $(z_1=z_2)$ deyiladi.

Demak haqiqiy qismlari o'zaro va mavhum qismlari o'zaro teng bo'lgan kompleks sonlar teng bo'lar ekan.

4-ta'rif. Ham haqiqiy qismi ham mavhum qismi noldan iborat kompleks son nolga teng deyiladi.

Demak, $a=0$, $b=0$ bo'lgandagina $z=0$ va aksincha $z=a+ib=0$ dan $a=0$, $b=0$ kelib chiqadi.

5-ta'rif. Faqat mavhum qismining ishorasi bilan farq qiluvchi ikkita $z=a+ib$ va $\overline{z}=a-ib$ kompleks sonlar o'zaro **qo'shma** kompleks sonlar deyiladi.

6-ta'rif. Haqiqiy va mavhum qismlarining ishoralari bilan farq qiluvchi ikkita

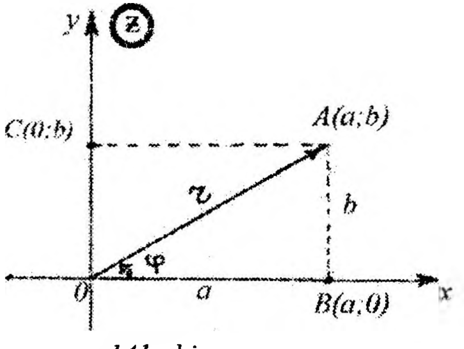
$$z_1=a+ib \quad \text{va} \quad z_2=-a-ib=-z_1$$

kompleks sonlar **qarama – qarshi** kompleks sonlar deyiladi.

28.3. Kompleks sonning geometrik tasviri

Har qanday $z=a+ib$ kompleks sonni $0xy$ tekislikda koordinatalari a va b bo'lgan $A(a,b)$ nuqta shaklida tasvirlash mumkin. Aksincha, $0xy$ tekislikdagi har qanday $A(a,b)$ nuqtaga bitta $z=a+ib$ kompleks son mos keladi.

Kompleks sonlar tasvirlanadigan tekislik z kompleks o'zgaruvchining tekisligi deyiladi va tekislikka doiracha ichiga z qo'yiladi (141-chizma).



141-chizma

Shunday qilib kompleks sonning geometrik tasviri ⓩ tekislikning nuqtasidan iborat ekan. Bunda $0x$ o'qda yotuvchi nuqtalar $z=a$ haqiqiy sonlarni $(b=0)$, $0y$ o'qda yotuvchi nuqtalar esa $z=bi$ sof mavhum sonlarni tasvirlaydi $(a=o)$. Shuning uchun kompleks sonlarni z kompleks o'zgaruvchining tekisligi ⓩ da tasvirlaganda $0x$

o‘q **haqiqiy o‘q**, *0y* o‘q **mavhum o‘q** deb ataladi. Umuman aytganda, kompleks sonlar to‘plami bilan (z) tekislikdagi barcha nuqtalar to‘plami orasida o‘zaro bir qiymatli moslik mavjud.

A(a,b) nuqtani koordinatalar boshi bilan tutashtirib $\overrightarrow{0A}$ vektorni hosil qilamiz. Ba‘zi hollarda $z=a+ib$ kompleks sonni $\overrightarrow{0A}$ vektor ko‘rinishda tasvirlash ma‘qul bo‘ladi. Bu ham kompleks sonning geometrik tasviri deyiladi.

Shunday qilib, kompleks sonning geometrik tasviri (z) tekislikdagi nuqtadan yoki vektordan iborat ekan.

28.4. Kompleks sonning trigonometrik shakli

Koordinatalar boshini qutb, *0x* o‘qning musbat yo‘nalishini qutb o‘qi deb (z) kompleks tekislikda qutb koordinatalar sistemasini kiritamiz. φ va r *A(a,b)* nuqtaning qutb koordinatalari bo‘lsin.

A nuqtaning qutb radiusi *r*, ya‘ni *A* nuqtadan qutbgacha bo‘lgan masofa $z=a+bi$ kompleks sonning **moduli** deyiladi va $|z|$ kabi belgilanadi.

Pifagor teoremasiga binoan 141-chizmadagi to‘g‘ri burchakli *OAB* uchburchakdan $r=\sqrt{a^2+b^2}$ kelib chiqadi. Masalan, $z_1=-3+4i$ sonning moduli $r_1=|z_1|=|-3+4i|=\sqrt{3^2+4^2}=5$ ga teng. Noldan farqli har qanday kompleks sonning moduli musbat haqiqiy sondir.

A nuqtaning qutb burchagi φ ni $z=a+bi$ kompleks sonning **argumenti** deyiladi va *Argz* kabi belgilanadi. Argument bir qiymatli aniqlanmay, balki $2\pi k$ qo‘shiluvchi qadar aniqlikda aniqlanadi, bunda k-butun son. Argumentning hamma qiymatlari orasidan $0\leq\varphi<2\pi$ tengsizlikni qanoatlantiruvchi bittasini tanlaymiz. Bu qiymat **bosh qiymat** deyiladi va $\varphi=\arg z$ kabi belgilanadi.

Dekart va qutb koordinatalari orasidagi bog‘lanish $a=r\cos\varphi$, $b=r\sin\varphi$ ni hisobga olib $z=a+bi=r\cos\varphi+ir\sin\varphi$ yoki

$$z=r(\cos\varphi+i\sin\varphi) \qquad (28.1)$$

tenglikka ega bo‘lamiz.

Bu tenglikning o'ng tomonidagi ifoda $z=a+bi$ kompleks sonning **trigonometrik shakldagi yozuvi** deb ataladi.

Qutb burchagi $\varphi=\text{arctg}\frac{b}{a}$ kabi topilishi ma'lum.

Shunday qilib, z kompleks sonning moduli deb uni tasvirlovchi vektorning uzunligiga, argumenti deb shu vektorning $0x$ o'qning musbat yo'nalishi bilan tashkil etgan burchagiga aytilar ekan.

$\varphi=\text{arctg}\frac{b}{a}$ argumentni hisoblashda z kompleks sonning koordinatalar tekisligining qaysi choragida yotishini hisobga olish kerak, chunki $\text{arctg}\frac{b}{a}$ qiymatga φ argumentning ikkita har xil qiymatlari mos keladi. Shuning uchun

$$\varphi=\arg z=\begin{cases}\text{arctg}\frac{b}{a}, & \text{agar } a>0,\ b>0 \text{ bo'lsa},\\ \pi+\text{arctg}\frac{b}{a}, & \text{agar } a<0,\ \text{b istalgan son bo'lsa},\\ 2\pi+\text{arctg}\frac{b}{a}, & \text{agar } a>0,\ b<0 \text{ bo'lsa}\end{cases}$$

tenglikdan foydalanish kerak. Masalan,

$arg(1+i)=\text{arctg}1=\frac{\pi}{4}$, chunki $a=1>0$, $b=1>0$, $arg(-1+i)=\pi+\text{arctg}(-1)=\pi-\frac{\pi}{4}=\frac{3\pi}{4}$, chunki $a=-1<0$, $b=1>0$, $arg(-1-i)=\pi+\text{arctg}1=\frac{5\pi}{4}$, chunki $a=-1<0$, $b=-1<0$, $arg(1-i)=2\pi+\text{arctg}(-1)==2\pi-\frac{\pi}{4}=\frac{7\pi}{4}$, chunki $a=1>0$, $b=-1<0$.

Kompleks sonning $z=a+bi$ ko'rinishdagi yozuvi kompleks sonning **algebraik** shakli deyiladi.

Kompleks son vektor shaklida tasvirlanganda haqiqiy songa $0x$ o'qda yotuvchi vektor, sof mavhum songa $0y$ o'qda yotuvchi vektor mos keladi.

1-misol. $z=a+bi$ va $\overline{z}=a-ib$ qo'shma kompleks sonlar bir xil modullarga ega va argumentlarining absolyut qiymatlari teng, ishoralari qarama-qarshi ekanligini ko'rsating.

Yechish. 142-chizmadan

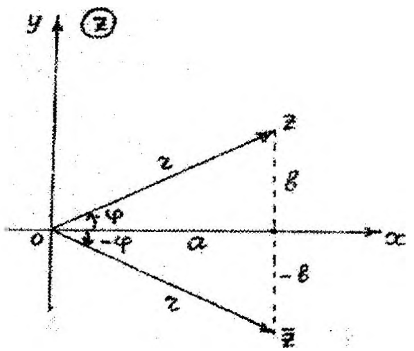
$$|z|=r=\sqrt{a^2+b^2} \text{ va } |\overline{z}|=r=\sqrt{a^2+b^2}$$

ekani, ya'ni $|z|=|\overline{z}|$ va $\arg z=-\arg\overline{z}$ ekani kelib chiqadi.

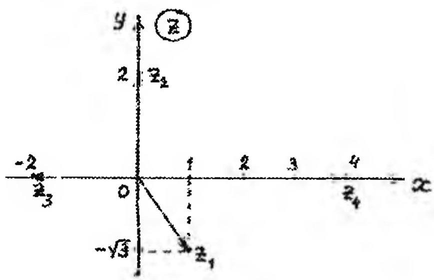
Izoh: Har qanday haqiqiy A sonni ham trigonometrik shaklda yozish mumkin, ya'ni $A>0$ bo'lsa, $A=A(\cos 0+i\sin 0)$,

$$A<0 \text{ bo'lsa, } A=|A|(\cos\pi+i\sin\pi) \qquad (28.2)$$

tengliklar o'rinlidir.



142-chizma.



143-chizma.

2-misol. $z_1=1-\sqrt{3}i$, $z_2=2i$, $z_3=-2$, $z_4=4$ kompleks sonlar trigonometrik shaklda yozilsin. (143-chizma)

Yechish. 1) $z_1=1-\sqrt{3}i$ son uchun $a=1$, $b=-\sqrt{3}$,

$r=\sqrt{1^2+\sqrt{3}^2}=2$,

$$\operatorname{tg}\varphi=-\frac{\sqrt{3}}{1}=-\sqrt{3},\quad \varphi=2\pi-\operatorname{arctg}\sqrt{3}=2\pi-\frac{\pi}{3}=\frac{5\pi}{3}.$$

Shunday qilib, $z_1=1-\sqrt{3}i=2\left(\cos\frac{5\pi}{3}+i\sin\frac{5\pi}{3}\right)$.

2) $z_2=2i$-sof mavhum son. $a=0,\ b=2,\ r=\sqrt{0^2+2^2}=2$,

$\varphi=\frac{\pi}{2},\quad z_2=2i=2\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$.

3) $z=-2$- manfiy haqiqiy son. Shuning uchun (28.2) formulaning ikkinchi tenglamasiga binoan $z_3=-2=|-2|(\cos\pi+i\sin\pi)$ bo'ladi.

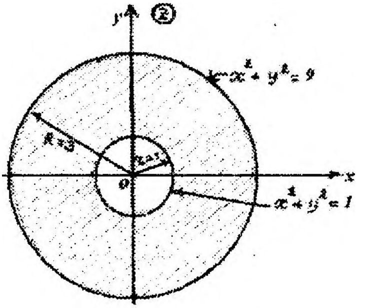
4) $z_4=4$- musbat haqiqiy son bo'lgani uchun (28.2) formulaning birinchi tenglamasiga binoan $z_4=4=4(\cos o+i\sin o)$ bo'ladi.

3-misol. $|z|\leq 3$ tengsizlikni qanoatlantiruvchi kompleks sonlarga mos Ⓩ–kompleks tekisligi nuqtalarining to'plami topilsin.

Yechish. $z=x+iy$ desak $|z|=\sqrt{x^2+y^2}$ bo'lib, berilgan tengsizlik $\sqrt{x^2+y^2}\leq 3$ yoki $x^2+y^2\leq 9$ ko'rinishga ega bo'ladi. $x^2+y^2=9$ tenglik markazi koordinatalar boshida bo'lib radiusi 3 ga teng aylanani ifodalaydi. Demak, $x^2+y^2\leq 9$-markazi koordinatalar boshida bo'lib, radiusi 3 ga teng doiraning nuqtalari. Bunda $x^2+y^2=9$ aylananing nuqtalari ham to'plamga tegishli.

4-misol. $1\leq|z|<3$ tengsizlikni qanoatlantiruvchi z kompleks sonlariga mos Ⓩ kompleks tekisligi nuqtalarining to'plami topilsin.

Yechish. 3-misolning natijasidan foydalanib $1\leq x^2+y^2<9$ tengsizliklarga ega bo'lamiz. $x^2+y^2\geq 1$ tengsizlik Ⓩ tekislikdagi markazi koordiatalar boshida bo'lib radiusi 1 ga teng aylanada va undan tashqarida yotgan nuqtalar to'plamini ifodalaydi. $x^2+y^2<9$ tengsizlik esa Ⓩ tekislikdagi markazi koordinatalar boshida bo'lib radiusi 3 ga teng aylananing ichida yotgan nuqtalar to'plamini ifodalaydi. Demak berilgan tengsizliklar Ⓩ tekislikdagi markazi koordinatalar boshida bo'lgan va radiuslari 1 ga va 3 ga teng konsentrik aylanalar orasidagi halqani ifodalar ekan. Bunda radiusi 1 ga teng aylananing nuqtalari ham halqaga tegishli (144-chizma).



144-chizma.

5-misol. $|z+2-i|=|z+4i|$ (b) tenglikni qanoatlantiruvchi z kompleks sonlar to'plami ⓩ kompleks tekisligida nimani ifodalaydi?

Yechish. $z=x+iy$ desak (b) tenglikni $|x+iy+2-i|=|x+iy+4i|$ yoki $|x+2+i(y-1)|=|x+i(y+4)|$ ko'rinishda yozish mumkin. Oxirgi tenglikni kompleks sonni modulini topish formulasiga asoslanib

$$\sqrt{(x+2)^2+(y-1)^2}=\sqrt{x^2+(y+4)^2} \qquad \text{(b)}$$

kabi yozamiz. Bu yerdagi $\sqrt{(x+2)^2+(y-1)^2}$ ifoda $z=x+iy$ kompleks songa mos keluvchi $A(x,y)$ nuqtadan $M(-2;1)$ nuqtagacha masofani, $\sqrt{(x^2+(y+4)^2}$ esa shu $A(x,y)$ nuqtadan $N(0;-4)$ nuqtagacha masofani ifodalaydi. Demak, *(b)* tenglik $A(x,y)$ nuqtadan $M(-2;1)$ va $N(0;-4)$ nuqtalargacha masofalar teng ekanligini ko'rsatadi. Kesmaning o'rta perpendikulyari uning uchlaridan bir xil masofada yotishini hisobga olsak berilgan tenglamadagi kompleks sonlarga ⓩ kompleks tekislikdagi MN kesmani o'rta perpendikulyarini ifodalovchi to'g'ri chiziqning nuqtalari to'plami mos kelishi ayon bo'ladi.

O'z-o'zini tekshirish uchun savollar

1. Kompleks son deb nimaga aytiladi?
2. Qanday kompleks sonlar teng deyiladi?
3. Qanday kompleks sonlar o'zaro qo'shma deyiladi?

4. Qanday kompleks sonlar qarama-qarshi kompleks sonlar deyiladi?
5. Kompleks sonning geometrik tasviri nimadan iborat?
6. Kompleks sonning moduli nima?
7. Kompleks sonning argumenti deb nimaga aytiladi?
8. Kompleks sonning trigonometrik shaklini yozing?
9. Kompleks sonning algebraik shakli bilan trigonometrik shakli orasida qanday bog'lanish mavjud?
10. Kompleks sonlar orasida katta va kichik tushunchalari mavjudmi?

Mustaqil yechish uchun mashqlar

1. *a)* $z=3$, *b)* $z=2i$, *c)* $z=-2$, *d)* $z=-3i$ kompleks sonlar (z) tekisligida vektor ko'rinishida tasvirlansin hamda ularning modullari va argumentlari aniqlansin.

2. Ushbu ifodalar trigonometrik shaklga keltirilsin:

a) $1+i$. Javob: $\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$.

b) $1-i$. Javob: $\sqrt{2}\left(\cos\frac{7\pi}{4}+i\sin\frac{7\pi}{4}\right)$.

c) $-1+i$. Javob: $\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$.

d) $-1-i$. Javob: $\sqrt{2}\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)$.

e) 3. Javob: $3(\cos o+i\sin o)$.

f) -4. Javob: $4(\cos\pi+i\sin\pi)$.

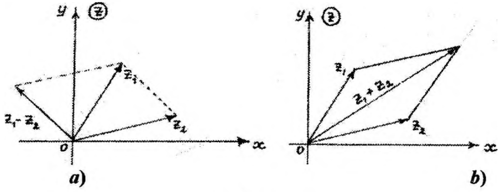
3. $|i-1+2z|\geq 9$ ni qanoatlantiruvchi z kompleks sonlar to'plami (z) kompleks tekisligida nimani ifodalaydi? Javob: Markazi $O_1\left(-\frac{1}{2};\frac{1}{2}\right)$ nuqtada va radiusi $R=4,5$ bo'lgan aylanada va undan tashqarida yotgan nuqtalar to'plamini ifodalaydi.

29. KOMPLEKS SONLAR USTIDA ASOSIY AMALLAR

29.1. Kompleks sonlarni qo'shish. Ikkita $z_1=a_1+ib_1$ va $z_2=a_2+ib_2$ kompleks sonlarning yig'indisi deb

$$z_1+z_2=(a_1+ib_1)+(a_2+ib_2)=(a_1+a_2)+i(b_1+b_2) \qquad (29.1)$$

tenglik bilan aniqlanuvchi kompleks songa aytiladi. (29.1) formuladan vektor bilan tasvirlangan kompleks sonlarni qo'shish-vektorlarni qo'shish qoidasiga muvofiq bajarilishi kelib chiqadi. (145^b-chizma)



145-chizma.

29.2 Kompleks sonlarni ayirish. Ikkita $z_1=a_1+ib_1$ va $z_2=a_2+ib_2$ kompleks sonlarning ayirmasi (z_1-z_2) deb shunday kompleks songa aytiladiki, unga z_2 kompleks sonni qo'shganda z_1 kompleks son hosil bo'ladi (145^a-chizma).

$$z_1-z_2=(a_1+ib_1)-(a_2+ib_2)=(a_1-a_2)+i(b_1-b_2). \qquad (29.2)$$

Ikki kompleks son ayirmasining moduli shu sonlarni ⓩ tekisligida tasvirlovchi $A(a_1;b_1)$ va $B(a_2;b_2)$ nuqtalar orasidagi masofaga teng:

$$|z_1-z_2|=\sqrt{(a_1-a_2)^2+(b_1-b_2)^2}.$$

1-misol. $z_1=3+2i$ va $z_2=2-i$ kompleks sonlarning yig'indisi va ayirmasini toping.

Yechish. $z_1+z_2=(3+2i)+(2-i)=(3+2)+i(2-1)=5+i,$

$z_1-z_2=(3+2i)-(2-i)=(3-2)+i(2-(-1))=1+3i.$

29.3. Kompleks sonlarni ko'paytirish. $z_1=a_1+ib_1$ va $z_2=a_2+ib_2$ kompleks sonlarning ko'paytmasi deb, $i^2=-1$ ekanligini hisobga olib bu sonlarni ikki hadlarni ko'paytirish qoidasi bo'yicha ko'paytirish natijasida hosil bo'lgan

$$z_1\cdot z_2=(a_1a_2-b_1b_2)+i(a_1b_2+b_1a_2) \qquad (29.3)$$

kompleks songa aytiladi.

z_1 va z_2 kompleks sonlar trigonometrik shaklda berilgan bo'lsin: $z_1=r_1(\cos\varphi_1+i\sin\varphi_1)$, $z_2=r_2(\cos\varphi_2+i\sin\varphi_2)$.

Shu kompleks sonlarning ko'paytmasini topamiz.

$$z_1\cdot z_2=r_1(\cos\varphi_1+i\sin\varphi_1)\cdot r_2(\cos\varphi_2+i\sin\varphi_2)=$$
$$=r_1\cdot r_2[\cos\varphi_1\cdot\cos\varphi_2+i\cos\varphi_1\sin\varphi_2++i\sin\varphi_1\cdot\cos\varphi_2+i^2\sin\varphi_1\cdot\sin\varphi_2]=$$
$$=r_1\cdot r_2[(\cos\varphi_1\cdot\cos\varphi_2-\sin\varphi_1\sin\varphi_2)+i(\cos\varphi_1\cdot\sin\varphi_2++\sin\varphi_1\cos\varphi_2)]=$$
$$r_1\cdot r_2[\cos(\varphi_1+\varphi_2)+i\sin(\varphi_1+\varphi_2)].$$

Shunday qilib,

$$z_1\cdot z_2=r_1\cdot r_2[\cos(\varphi_1+\varphi_2)+i\sin(\varphi_1+\varphi_2)], \qquad (29.4)$$

ya'ni ikkita trigonometrik shakilda berilgan kompleks sonlar ko'paytirilganda ularning modullari ko'paytiriladi, argumentlari esa qo'shiladi.

2-misol. $z_1=3-i$ va $z_2=4+2i$ kompleks sonlarning ko'paytmasi topilsin.

Yechish. $z_1\cdot z_2=(3-i)(4+2i)=12+6i-4i-2i^2=14+2i.$

3-misol. $z_1=4\left(\cos\frac{11}{6}\pi+i\sin\frac{11}{6}\pi\right)$ va

$$z_2=3\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$$

kompleks sonlarning ko'paytmasi topilsin.

Yechish. (29.4) formulaga binoan:

$$z_1z_2=4\left(\cos\frac{11}{6}\pi+i\sin\frac{11}{6}\pi\right)\times3\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)=$$

$$=4\times3\left[\cos\left(\frac{11}{6}\pi+\frac{\pi}{3}\right)+i\sin\left(\frac{11}{6}\pi+\frac{\pi}{3}\right)\right]=$$

$$=12\left[\cos\left(2\pi+\frac{\pi}{6}\right)+i\sin\left(2\pi+\frac{\pi}{6}\right)\right]=$$

$$=12\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)=12\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)=6\sqrt{3}+6i.$$

4-misol. $z=a+ib$ va $\bar{z}=a-ib$ qo‘shma kompleks sonlar ko‘paytirilsin. **Yechish.** $z\cdot\bar{z}=(a+ib)(a-ib)=a^2-(ib)^2=a^2+b^2$ yoki $z\cdot\bar{z}=|z|^2$, chunki $|z|=|\bar{z}|=\sqrt{a^2+b^2}$.

Demak, qo‘shma kompleks sonlarni ko‘paytmasi haqiqiy son ekan.

29.4 Kompleks sonlarni bo‘lish. $z_1=a_1+ib_1$ sonning $z_2=a_2+ib_2$ $(a_2^2+b_2^2\neq0)$ kompleks soniga bo‘linmasi deb z_2 son bilan ko‘paytmasi z_1 ga teng $z=x+iy$ kompleks songa aytiladi. Demak $z=\frac{z_1}{z_2}$ va $z_1=z\cdot z_2$ tengliklar teng kuchli.

Kompleks sonni kompleks songa bo‘lish amali bo‘linuvchi va bo‘luvchini bo‘luvchining qo‘shmasiga ko‘paytirish natimjasida amalga oshiriladi:

$$\frac{z_1}{z_2}=\frac{a_1+ib_1}{a_2+ib_2}=\frac{(a_1+ib_1)(a_2-ib_2)}{(a_2+ib_2)(a_2-ib_2)}=\frac{a_1a_2+b_1b_2}{a_2^2+b_2^2}+i\frac{b_1a_2-a_1b_2}{a_2^2+b_2^2}.$$

Agar kompleks sonlar $z_1=r_1(\cos\varphi_1+i\sin\varphi_1)$ va $z_2=r_2(\cos\varphi_2+i\sin\varphi_2)$ trigonometrik shaklda berilgan bo‘lsa, u holda:

$$\frac{z_1}{z_2}=\frac{r_1(\cos\varphi_1+i\sin\varphi_1)}{r_2(\cos\varphi_2+i\sin\varphi_2)}=\frac{r_1(\cos\varphi_1+i\sin\varphi_1)\cdot(\cos\varphi_2-i\sin\varphi_2)}{r_2(\cos\varphi_2+i\sin\varphi_2)\cdot(\cos\varphi_2-i\sin\varphi_2)}=$$

$$=\frac{r_1[(\cos\varphi_1\cos\varphi_2+\sin\varphi_1\sin\varphi_2)+i(\sin\varphi_1\cos\varphi_2-\sin\varphi_2\cos\varphi_1)]}{r_2(\cos^2\varphi_2-(i\sin\varphi_2)^2)}=$$

$$=\frac{r_1[\cos(\varphi_1-\varphi_2)+i\sin(\varphi_1-\varphi_2)]}{r_2(\cos^2\varphi_2+\sin^2\varphi_2)}=\frac{r_1}{r_2}[\cos(\varphi_1-\varphi_2)+i\sin(\varphi_1-\varphi_2)].$$

Shunday qilib,

$$\frac{z_1}{z_2}=\frac{r_1}{r_2}[\cos(\varphi_1-\varphi_2)+i\sin(\varphi_1-\varphi_2)], \tag{29.5}$$

ya'ni ikkita trigonometrik shakldagi kompleks sonlarni bo'lishda bo'linuvchining moduli bo'luvchining moduliga bo'linadi, bo'linuvchining argumentidan bo'luvchining argumenti ayriladi.

5-misol. $z_1=3-2i$ kompleks son $z_2=4+i$ songa bo'linsin.

Yechish.

$$\frac{z_1}{z_2}=\frac{3-2i}{4+i}=\frac{(3-2i)(4-i)}{(4+i)(4-i)}=\frac{3\cdot4-2-i(2\cdot4+3\cdot1)}{4^2+1^2}=\frac{10-i\cdot11}{17}=\frac{10}{17}-\frac{11}{17}i.$$

6-misol. $z_1=4\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$ kompleks son $z_2=2\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$ songa bo'linsin.

Yechish. (29.5) formulaga binoan:

$$\frac{z_1}{z_2}=\frac{4}{2}\left[\cos\left(\frac{3\pi}{4}-\frac{\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}-\frac{\pi}{4}\right)\right]=2\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=2(0+i)=2i.$$

29.5. Kompleks sonni darajaga ko'tarish. Kompleks sonlarni ko'paytirish qoidasiga ko'ra *n* ta

$$z_1=r_1(\cos\varphi_1+i\sin\varphi_1),$$

$$z_2=r_2(\cos\varphi_2+i\sin\varphi_2),$$

$$\dots\dots\dots\dots\dots$$

$$z_n=r_n(\cos\varphi_n+i\sin\varphi_n)$$

sonlarning ko‘paytmasi

$z_1 z_2 \ldots z_n = r_1 \cdot r_2 \ldots \cdot r_n[\cos(\varphi_1+\varphi_2+\ldots+\varphi_n)+i\sin(\varphi_1+\varphi_2+\ldots+\varphi_n)]$ bo‘lishi ayon.

Bundan $z_1=z_2=\ldots z_n=z=r(\cos\varphi+i\sin\varphi)$ bo‘lganda

$$z^n=[r(\cos\varphi+i\sin\varphi)]^n=r^n(\cos n\varphi+i\sin n\varphi) \qquad (29.6)$$

formulaga ega bo‘lamiz.

Bu formula **Muavr formulasi** deb ataladi. Bu formula trigonometrik shakilda berilgan kompleks sonni biror natural darajaga ko‘tarish uchun uning modulini shu darajaga ko‘tarish lozimligini, argumentini esa daraja ko‘rsatgichiga ko‘paytirish kerakligini ko‘rsatadi.

7-misol. $(1+i)^{20}$ ni hisoblang.

Yechish. $|z|=r=\sqrt{1^2+1^2}=\sqrt{2}$ $\quad \varphi=\mathrm{arctg}\frac{1}{1}=\frac{\pi}{4}$ bo‘lgani uchun

$1+i=z=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$ bo‘lib (29.6) formulaga binoan

$$z^{20}=(1+i)^{20}=$$

$$=\left[\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]^{20}=\sqrt{2}^{20}\left(\cos 20\cdot\frac{\pi}{4}+i\sin 20\cdot\frac{\pi}{4}\right)=$$

$$=2^{10}(\cos 5\pi+i\sin 5\pi)=1024(\cos\pi+i\sin\pi)=-1024.$$

Muavr formulasida $r=1$ deb olinsa

$$(\cos\varphi+i\sin\varphi)^n=\cos n\varphi+i\sin n\varphi \qquad (29.7)$$

formula kelib chiqadi. Bu formula $\cos n\varphi$, $\sin n\varphi$ funksiyalarni $\cos\varphi$, $\sin\varphi$ funksiyalarning darajalari orqali ifodalash imkonini beradi.

Masalan, n=2 da $(\cos\varphi+i\sin\varphi)^2=\cos 2\varphi+i\sin 2\varphi$ ga ega bo‘lamiz, bundan:

$$\cos^2\varphi+2i\cos\varphi\sin\varphi+i^2\sin^2\varphi=\cos 2\varphi+i\sin 2\varphi,$$

$$\cos^2\varphi-\sin^2\varphi+2i\sin\varphi\cos\varphi=\cos 2\varphi+i\sin 2\varphi.$$

Ikki kompleks sonlarni tengligi shartidan foydalansak

$\cos 2\varphi=\cos^2\varphi-\sin^2\varphi$, $\sin 2\varphi=2\sin\varphi\cdot\cos\varphi$ ma‘lum formulalarga ega bo‘lamiz.

Shuningdek $n=3$ da (29.7) formula $(\cos\varphi+i\sin\varphi)^3=\cos3\varphi+i\sin3\varphi$ ko'rinishga ega bo'lib, bundan:

$$\cos^3\varphi+3\cdot\cos^2\varphi\cdot i\sin\varphi+3\cos\varphi(i\sin\varphi)^2+(i\sin\varphi)^3=\cos3\varphi+i\sin3\varphi,$$
$$(\cos^3\varphi-3\cdot\cos\varphi\cdot\sin^2\varphi)+i(3\cos^2\varphi\sin\varphi-\sin^3\varphi)=\cos3\varphi+i\sin3\varphi.$$

Ikki kompleks sonlarni tengligi shartiga asoslanib

$$\cos3\varphi=\cos^3\varphi-3\cos\varphi\cdot\sin^2\varphi,\quad \sin3\varphi=3\cos^2\varphi\sin\varphi-\sin^3\varphi$$

formulalarni hosil qilamiz.

29.6 Kompleks sondan ildiz chiqarish. z kompleks sonni n –darajali ildizi $\sqrt[n]{z}$ deb n –darajasi ildiz ostidagi z songa teng bo'lgan w kompleks songa aytiladi, ya'ni $w^n=z$ bo'lganda $\sqrt[n]{z}=w$ $(n\in N)$.

Agar $z=r(\cos\varphi+i\sin\varphi)$ va $w=\rho(\cos\psi+i\sin\psi)$ bo'lsa

$$\sqrt[n]{r(\cos\varphi+i\sin\varphi)}==\rho(\cos\psi+i\sin\psi)$$

tenglik o'rinlidir. Bundan tenglikning ikkala tomonini n-darajaga ko'tarib keyin Muavr formulasidan foydalansak

z $r(\cos\varphi+i\sin\varphi)=[\rho(\cos\psi+i\sin\psi)]^n=\rho^n(\cos n\psi+i\sin n\psi)$ hosil bo'ladi.

Teng kompleks sonlarni modullari teng, argumentlari esa 2π ga karrali songa farq qilishini hisobga olsak oxirgi tenglikdan

ρ^n $r, n\psi$ $\varphi+2\pi k$ ga ega bo'lamiz. Bundan ρ va ψ ni topamiz:

$$\rho=\sqrt[n]{r},\qquad \Psi=\frac{\varphi+2\pi k}{n},$$

bunda k -istalgan butun son, $\sqrt[n]{r}$ -arifmetik ildiz.

Shunday qilib,

$$w_{k+1}=\sqrt[n]{r(\cos\varphi+i\sin\varphi)}=\sqrt[n]{r}\left(\cos\frac{\varphi+2\pi k}{n}+i\sin\frac{\varphi+2\pi k}{n}\right).\quad(29.8)$$

tenglikka ega bo'ldik, bu yerdagi k ga o dan $n-1$ gacha qiymatlarini berib, ildizning n ta har xil qiymatlarini topamiz. k ning $n-1$ dan katta qiymatlarida argumentlar topilgan qiymatlardan 2π ga karrali songa farq qiladi va demak, topilgan ildizlar avvalgilar bilan bir xil bo'ladi. Masalan, $k=0$ va $k=n$ bo'lgangdagi, $k=1$ va $k=n+1$ bo'lgandagi va hokazo ildizlar bir xil bo'ladi.

Shunday qilib, kompleks sonning n-darajali ildizi n ta har xil qiymatlarga ega bo'lar ekan.

Shuningdek noldan farqli haqiqiy sonning n-darajali ildizi ham n ta har xil qiymatlarga ega bo'ladi, chunki haqiqiy son kompleks sonning xususiy holi.

8-misol. $\sqrt[3]{1}$ ning barcha qiymatlari topilsin va ular ⓩ kompleks tekislikda vektor shaklida tasvirlansin.

Yechish. $z=1=1+0\cdot i$ ni trigonometrik shaklda yozamiz. $a=1$, $b=0$ bo'lgani uchun $|z| = r = \sqrt{1^2 + 0^2} = 1$, $\varphi = \operatorname{arctg}\frac{0}{1} = 0$ va $z = \cos 0 + i\sin 0$ ga ega bo'lamiz.

U holda (29.8) formula

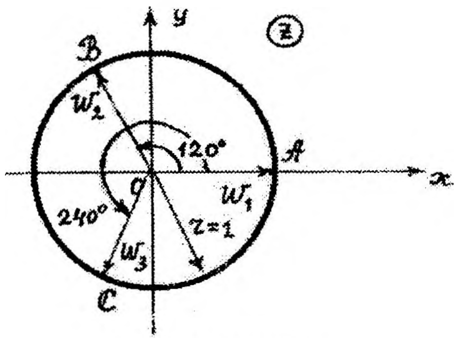
$\sqrt[3]{1} = \sqrt[3]{\cos 0 + i\sin 0} = \cos\frac{2\pi k}{3} + i\sin\frac{2\pi k}{3}$ ko'rinishga ega bo'ladi, bunda $\kappa=0,1,2$.

$\kappa=0$ da $w_1=\cos 0+i\sin 0=1$,

$$k = 1\text{da } w_2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \cos\left(\frac{\pi}{2}+\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{2}+\frac{\pi}{6}\right) = -\sin\frac{\pi}{6} + i\cos\frac{\pi}{6} = -\frac{1}{2} + i\frac{\sqrt{3}}{2},$$

$$k = 2\text{da } w_3 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = \cos\left(\pi+\frac{\pi}{3}\right) + i\sin\left(\pi+\frac{\pi}{3}\right) = -\cos\frac{\pi}{3} - i\sin\frac{\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

w_1, w_2 va w_3 kompleks sonlarning barchasini moduli 1 ga teng ekanligini hisobga olib markazi koordinatalar boshida bo'lib radiusi 1 ga teng aylana yasaymiz. Boshi koordinatalar boshida bo'lib uchi shu aylanada yotgan, hamda $0x$ o'qning musbat yo'nalishi bilan $0^0, 120^0$ va 240^0 $\left(0, \frac{2\pi}{3} \, va \, \frac{4\pi}{3}\right)$ burchak tashkil etuvchi $\overrightarrow{OA}$, $\overrightarrow{OB}$ va $\overrightarrow{OC}$ vektorlar mos ravishda w_1, w_2 va w_3 kompleks sonlarining geometrik tasviri bo'ladi (146-chizma).



146-chizma.

29.7. Ikki hadli tenglamalarni yechish. $z^n=A$ ko'rinishdagi tenglama **ikki hadli tenglama** deyiladi, bunda A aniq kompleks son. Shu tenglamaning ildizlarini topamiz.

a) A kompleks son bo'lsin. Bu holda (29.8) formulaga binoan tenglamaning ildizlari

$$z_{k+1}=\sqrt[n]{A}=\sqrt[n]{|A|}\left(\cos\frac{\varphi+2k\pi}{n}+i\sin\frac{\varphi+2k\pi}{n}\right) \qquad (29.9)$$

formula yordamida topiladi, bunda $\varphi=\arg A$, $k=0.1,2,\ldots,n-1$.

b) A musbat haqiqiy son bo'lsin. U holda $\varphi=\arg A=0$ bo'lib (29.9) formula

$$z_{k+1}=\sqrt[n]{A}\cdot\left(\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}\right) \qquad (29.10)$$

ko'rinishini oladi $(k=0,1,2,,\ldots,n-1)$.

d) A manfiy haqiqiy son bo'lsin. U holda $\varphi=\arg A=\pi$ bo'lganligi sababli (29.9) formuladan

$$z_{k+1}=\sqrt[n]{|A|}\left(\cos\frac{\pi+2k\pi}{n}+i\sin\frac{\pi+2k\pi}{n}\right) \qquad (29.11)$$

hosil bo'ladi.

Xususiy holda $z^n=1$ tenglamaning barcha ildizlari

$$z_{k+1}=\sqrt[n]{1}=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n} \qquad (29.12)$$

formula yordamida, $z^n=-1$ tenglamaning barcha ildizlari

$$z_{k+1}=\sqrt[n]{-1}=\cos\frac{\pi+2k\pi}{n}+i\sin\frac{\pi+2k\pi}{n} \qquad (29.13)$$

formula yordamida topiladi $(\kappa=0,1,2,...n-1)$.

9-misol. $z^4=1$ tenglama yechilsin.

Yechish. (29.12) formulaga binoan

$$z_{k+1}=\cos\frac{2k\pi}{4}+i\sin\frac{2k\pi}{4}=\cos\frac{k\pi}{2}+i\sin\frac{k\pi}{2}$$

bo'ladi. k o'rniga $0,1,2,3$ qiymatlarni qo'yib ushbularni topamiz:

$$z_1=\cos 0+i\sin 0=1,$$

$$z_2=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}=0+i=i,$$

$$z_3=\cos\pi+i\sin\pi=-1,$$

$$z_4=\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}=-i.$$

O'z-o'zini tekshirish uchun savollar

1. Algebraik shakldagi kompleks sonlarni qo'shish, ayirish, ko'paytirish va bo'lish qoidalari qanday?
2. Kompleks sonlar ayirmasining moduli nimani ifodalaydi?
3. Trigonometrik shakldagi kompleks sonlarni ko'paytirish va bo'lish formulalarini yozing.
4. O'zaro qo'shma kompleks sonlarni ko'paytmasi qanaqa son bo'ladi?
5. Muavr formulasini yozing.
6. Trigonometrik shakldagi kompleks sonni n- darajaga ko'tarish uchun nima qilinadi?
7. Trigonometrik shakldagi kompleks sonning barcha n-darajali ildizlarini topish formulasini yozing.
8. Manfiy sondan kvadrat ildiz chiqadimi?
9. Ikki hadli tenglamaning ildizlari qanday topiladi?
10. $x^n=A$ ko'rinishdagi tenglama nechta har xil ildizlarga ega?

Mustaqil yechish uchun mashqlar

1. $(3+2i)+(2-i)$ topilsin. Javob: $5+i$.
2. $(4+3i)-(6-4i)$ topilsin. Javob: $-2+7i$.
3. $(3+2i)(2-3i)$ topilsin. Javob: $12-5i$.
4. $(3+5i)(4-i)$ topilsin. Javob: $17+17i$.
5. $\frac{3-i}{4+5i}$ topilsin. Javob: $\frac{7}{41}-\frac{19}{41}i$.
6. $\frac{1-i}{-2-2i}$ topilsin. Javob: $\frac{1}{2}i$.
7. $(4-7i)^3$ topilsin. Javob: $-524+7i$.
8. $(2(\cos 18^\circ+i\sin 18^\circ))^5$ topilsin. Javob: $32i$.
9. Quyidagi algebraik shakldagi kompleks sonlarni trigonometrik shaklga keltirib, so'ngra Muavr formulasini qo'llang.

a) $(1+i)^{10}$; b) $(1-i)^{16}$; d) $(\sqrt{3}+i)^{20}$; e) $(\sqrt{3}-i)^{30}$;

f) $(1+\cos\alpha+i\sin\alpha)^n$.

10. $z_1=3(\cos 20^\circ-i\sin 20^\circ)$ son $z_2=2(\cos 10^\circ-i\sin 10^\circ)$ songa bo'linsin.

Javob: $\frac{3}{4}\left(\sqrt{3}-i\right)$.

11. $\sqrt[3]{-8}$ topilsin. Javob: $1+i\sqrt{3}$; -2; $1-i\sqrt{3}$.

12. $z_k=\sqrt[3]{-\sqrt{2}+i\sqrt{2}}$ topilsin. Javob: $z_1=\sqrt[3]{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$,

$z_2=\sqrt[3]{2}\left(\cos\frac{11\pi}{12}+i\sin\frac{11\pi}{12}\right)$, $z_3=\sqrt[3]{2}\left(\cos\frac{19\pi}{12}+i\sin\frac{19\pi}{12}\right)$.

13. $z^3=i$ tenglama yechilsin.

Javob: $z_1=\frac{\sqrt{3}}{2}+i\cdot\frac{1}{2}$; $z_2=-\frac{\sqrt{3}}{2}+i\frac{1}{2}$; $z_3=-i$.

30. KOMPLEKS O'ZGARUVCHINING FUNKSIYASI

30.1. Ko'rsatkichi kompleks bo'lgan ko'rsatkichli funksiya va uning xossalari

x va y haqiqiy o'zgaruvchilar bo'lganda $z=x+iy$ kompleks o'zgaruvchi deyiladi.

Ta'rif. Agar kompleks o'zgaruvchi $z=x+iy$ ning biror kompleks qiymatlar sohasidagi har bir qiymatiga biror usul yoki qoida yordamida boshqa w kompleks o'zgaruvchining aniq qiymati mos kelsa, w kompleks o'zgaruvchi z ning funksiyasi deyiladi va $w=f(z)$ yoki $w=w(z)$ kabi belgilanadi.

Biz kompleks o'zgaruvchining funksiyalari nazariyasini o'rganishni keyinga qoldirib uning faqat bitta funksiyasi-ko'rsakichli funksiyasi

$$w=e^{z} \quad \text{yoki} \quad w=e^{x+iy}$$

ni qaraymiz. Bu funksiya

$$e^{x+iy}=e^{x}\cdot e^{iy}=e^{x}(\cos y+i\sin y) \qquad (30.1)$$

tenglik yordamida aniqlanadi. Demak

$$w(z)=e^{x}(\cos y+i\sin y).$$

(30.1) formulaga asoslanib

$$e^{1+\frac{\pi}{4}i}=e\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)=e\left(\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}\right)=\frac{e\sqrt{2}}{2}+i\frac{e\sqrt{2}}{2},$$

$$e^{0+\frac{\pi}{2}i}=e^{0}\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=i,$$

$$e^{\frac{3\pi}{2}i}=e^{0}\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)=-i.$$

tengliklarga ega bo'lamiz.

Kompleks o'zgaruvchining ko'rsatkichli funksiyasi quyidagi xossalarga ega.

1. Har qanday z_1, z_2 kompleks sonlar uchun

$$e^{z_1+z_2}=e^{z_1}\cdot e^{z_2} \qquad (30.2)$$

tenglik o‘rinli.

Isboti. $z_1=x_1+iy_1$, $z_2=x_2+iy_2$ bo‘lsin. U holda:

$$e^{z_1+z_2}=e^{x_1+iy_1+x_2+iy_2}=e^{(x_1+x_2)+i(y_1+y_2)}=$$

$$=e^{x_1+x_2}[\cos(y_1+y_2)+i\sin(y_1+y_2)]. \qquad (30.3)$$

Ikkinchi tomonidan trigonometrik shakldagi kompleks sonlarni ko‘paytirish qoidasiga asosan:

$$e^{z_1}\cdot e^{z_2}=e^{x_1+iy_2}e^{x_2+iy_2}=e^{x_2}\left[\cos y_1+i\sin y_1\right]\cdot e^{x_2}$$

$$\left[\cos y_2+i\sin y_2\right]=e^{x_1+x_2}\left[\cos(y_2+y_2)+i\sin(y_2+y_2)\right] \qquad (30.4).$$

(30.3) va (30.4) tengliklarni o‘ng tomonlari teng, demak, chap tomonlari ham teng bo‘ladi: $e^{z_1+z_2}=e^{z_1}\cdot e^{z_2}$.

2. Istalgan z_1, z_2 kompleks sonlar uchun

$$\frac{e^{z_1}}{e^{z_2}}=e^{z_1-z_2} \qquad (30.5)$$

tenglik o‘rinli.

Bu xossaning isboti 1-xossaning isbotiga o‘xshaganligi uchun uning isbotini o‘quvchiga qoldiramiz.

3. Ixtiyoriy m butun son uchun

$$(e^z)^m=e^{mz} \qquad (30.6)$$

tenglik o‘rinli.

Isboti. $z=x+iy$ bo‘lsin. U holda

$$(e^z)^m=(e^{x+iy})^m=(e^x e^{iy})^m=e^{mx}e^{imy}=e^{mx+imy}=e^{m(x+iy)}=e^{mz}.$$

4. Ixtiyoriy z uchun

$$e^{z+2\pi i}=e^z \qquad (30.7)$$

tenglik o‘rinli.

Haqiqatan (30.2)va (30.1) formulalarga asosan:

$$e^{z+2\pi i}=e^z\cdot e^{2\pi i}=e^z(\cos 2\pi+i\sin 2\pi)=e^z.$$

(30.7) ayniyatdan e^z ko‘rsatkichli funksiya davri $2\pi i$ ga teng davriy funksiya ekanligi ravshan.

30.2. Haqiqiy o'zgaruvchining kompleks funksiyasini limiti va hosilasi

$u(x)$, $v(x)$ funksiyalar x haqiqiy o'zgaruvchining haqiqiy funksiyalari bo'lsin.

$$W=u(x)+iv(x)$$

haqiqiy o'zgaruvchining kompleks funksiyasini qaraymiz.

a) Agar $\lim\limits_{x\to x_0} u(x)=u_o$, $\lim\limits_{x\to x_0} v(x)=v_o$

chekli limitlar mavjud bo'lsa

$$w_o=u_o+iv_o$$

kompleks son W kompleks o'zgaruvchining $x\to x_0$ dagi **limiti** deyiladi.

b) Agar $u'(x)$ va $v'(x)$ hosilalar mavjud bo'lsa,

$$W'_x=u'(x)+iv'(x). \qquad (30.8)$$

ifoda haqiqiy o'zgaruvchining kompleks funksiyasi W ning haqiqiy o'zgaruvchi x bo'yicha **hosilasi** deyiladi.

Endi $W=e^{kx}$ ko'rsatkichli funksiyaning w'_x hosilani topamiz, bunda $k=\alpha+\beta i$ va α, β o'zgarmas haqiqiy sonlar. (30.1) formulaga muvofiq

$$W=e^{kx}=e^{(\alpha+\beta i)x}=e^{\alpha x}e^{\beta xi}=e^{\alpha x}(\cos\beta x+i\sin\beta x)$$

yoki

$$W=e^{\alpha x}\cos\beta x+ie^{\alpha x}\sin\beta x.$$

(30.8.) formulaga asosan:

$$W_x'=(e^{\alpha x}\cos\beta x)'+i(e^{\alpha x}\sin\beta x)'=e^{\alpha x}\alpha\cos\beta x-e^{\alpha x}\beta\sin\beta x+$$
$$+i(e^{\alpha x}\alpha\sin\beta x+e^{\alpha x}\beta\cos\beta x)=e^{\alpha x}(\alpha\cos\beta x-$$
$$-\beta\sin\beta x)+ie^{\alpha x}(\alpha\sin\beta x+\beta\cos\beta x)=\alpha[e^{\alpha x}(\cos\beta x+i\sin\beta x)]+$$
$$+\beta i[e^{\alpha x}(\cos\beta x-\frac{1}{i}\sin\beta x)]=\alpha e^{\alpha x+i\beta x}+\beta ie^{\alpha x+i\beta x}=(\alpha+\beta i)e^{(\alpha+\beta i)x}$$

chunki $\frac{1}{i}=-i$.

Shunday qilib, k o'zgarmas kompleks son bo'lganda ham

$(e^{(\alpha+i\beta)x})' = (\alpha + i\beta)\cdot e^{(\alpha+i\beta)x}$ yoki $(e^{kx})' = ke^{kx}$ ko'rsatkichli funksiyani differensiallashning odatdagi formulasiga ega bo'ldik. Shuningdek

$$(e^{kx})'' = ((e^{kx})')' = (ke^{kx})' = k^2e^{kx},$$

$$(e^{kx})''' = ((e^{kx})'')' = (k^2e^{kx})' = k^3e^{kx}$$

va hokazo har qanday natural n son uchun

$$(e^{kx})^{(n)} = k^n e^{kx}$$

tenglikka ega bo'lamiz.

Bu formulalar bizga keyinchalik ham, aniqrog'i differensial tenglamalarni qaraganda ham kerak bo'ladi.

30.3. Eyler formulasi. Kompleks sonning ko'rsatkichli shakli

(30.1) formulada $x=0$ desak

$$e^{iy}=\cos y+i\sin y \qquad (30.9)$$

formula hosil bo'ladi. Bu formula **Eyler formulasi** deb ataladi.

Eyler formulasida y ni $-y$ ga almashtirib $\cos y$ funksiyani juft, $\sin y$ toq funksiya ekanligini hisobga olsak

$$e^{-iy} = \cos y - i\sin y \qquad (30.10)$$

tenglik hosil bo'ladi. (30.9) va (30.10) ni hadma-had qo'shib

$e^{iy} + e^{-iy} = 2\cos y$ yoki bundan $\cos y = \dfrac{e^{iy} + e^{-iy}}{2}$ formulaga,

(30.9) dan (30.10) ni hadma-had ayirib

$e^{iy} - e^{-iy} = 2i\sin y$ yoki bundan

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i}$$

formulaga ega bo'lamiz.

Shunday qilib, trigonometrik funksiyalar bilan ko'rsatkichli funksiyalarni bog'lovchi formulaga ega bo'ldik.

1-misol. $A = \sin x + \sin 2x + \sin 3x + \ldots + \sin nx$ yig'indi topilsin.

Yechish. $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$ formuladan foydalanamiz.

U holda

$$A = \frac{e^{ix} - e^{-ix}}{2i} + \frac{e^{2ix} - e^{-2ix}}{2i} + \frac{e^{3ix} - e^{-3ix}}{2i} + \ldots + \frac{e^{nix} - e^{-nix}}{2i} =$$

$$= \frac{1}{2i}\Big[(e^{ix} + e^{2ix} + e^{3ix} + \ldots + e^{nix}) -$$

$$-(e^{-ix} + e^{-2ix} + e^{-3ix} + \ldots + e^{-nix})\Big] = \frac{1}{2i}\Big[e^{ix}(1 + e^{ix} + (e^{ix})^2 + \ldots + (e^{ix})^{n-1}) -$$

$$-e^{-ix}(1 + e^{-ix} + (e^{-ix})^2 + \ldots + (e^{-ix})^{n-1}\Big] = \frac{1}{2i}\left[\frac{e^{ix}(1 - e^{nix})}{1 - e^{ix}} - \frac{e^{-ix}(1 - e^{-nix})}{1 - e^{-ix}}\right]$$

bo'ladi. Biz bu yerda birinchi va ikkinchi qavs ichidagi ifodalarni mos ravishda maxraji e^{ix} va e^{-ix} bo'lgan geometrik progressiya deb qarab uning n ta hadlarining yig'indisini topish formulasi

$$1 + q + q^2 + \ldots + q^{n-1} = \frac{1 - q^n}{1 - q}$$

dan foydalandik.

Shunday qilib

$$A = \frac{1}{2i}\left[\frac{e^{ix}(1 - e^{nix})}{1 - e^{ix}} - \frac{e^{-ix}(1 - e^{-nix})}{1 - e^{-ix}}\right]$$

yoki

$$A = \frac{1}{2i}\left[\frac{1 - e^{nix}}{e^{-ix} - 1} - \frac{1 - e^{-nix}}{e^{ix} - 1}\right]$$

tenglikka ega bo'ldik. Bu yerda qavs ichidagi birinchi kasrning surat va maxrajini e^{-ix} ga ikkinchi kasrnikini e^{ix} ga ko'paytirildi.

Eyler formulasiga asosan

$$A=\frac{1}{2i}\left[\frac{1-(\cos nx+i\sin nx)}{\cos x-i\sin x-1}-\frac{1-(\cos nx-i\sin nx)}{\cos x+i\sin x-1}\right]=$$

$$=\frac{1}{2i}\left[\frac{(1-\cos nx-i\sin nx)}{(\cos x-1)-i\sin x}-\frac{(1-\cos nx)+i\sin nx}{(\cos x-1)+i\sin x}\right]=$$

$$=\frac{1}{2i}\left[\frac{((1-\cos nx)-i\sin nx)((\cos x-1)+i\sin x)-((1-\cos nx)+i\sin nx)((\cos x-1)-i\sin x)}{((\cos x-1)-i\sin x)((\cos x-1)+i\sin x)}\right]=$$

$$=\frac{1}{2i}\left[\frac{(1-\cos nx)(\cos x-1)+\sin nx\sin x-i(\sin nx(\cos x-1)+(1-\cos nx)\sin x)}{(\cos x-1)^2+\sin^2 x}-\right.$$

$$\left.-\frac{(1-\cos nx)(\cos x-1)+\sin nx\sin x+i(\sin nx(\cos x-1)-(1-\cos nx)\sin x)}{(\cos x-1)^2+\sin^2 x}\right]=$$

$$\frac{1}{2i}\frac{-2i\sin nx(\cos x-1)+2i(1-\cos nx)\sin x}{\cos^2 x-2\cos x+1+\sin^2 x}=$$

$$=\frac{-\sin nx\cos x+\sin nx+\sin x-\cos nx\sin x}{2-2\cos x}=$$

$$=\frac{\sin nx+\sin x-(\sin nx\cos x+\cos nx\sin x)}{2(1-\cos x)}$$

yoki $\sin\alpha+\sin\beta=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$ formulaga ko'ra $\sin nx+\sin x=$ $=2\sin\frac{n+1}{2}x\cos\frac{n-1}{2}x$, $1-\cos x=2\sin^2\frac{x}{2}$ va ikki Argument yig'indisining sinusini topish formulasiga ko'ra $\sin nx\cos x+\cos nx\sin x=\sin(n+1)x$ bo'lishini va $\sin 2\alpha=2\sin\alpha\cos\alpha$ formulaga binoan $\sin(n+1)x=2\sin\frac{n+1}{2}x\cdot\cos\frac{n+1}{2}x$ bo'lishini hisobga olsak

$$A=\frac{2\sin\frac{n+1}{2}\cdot\cos\frac{n-1}{2}x-\sin(n+1)x}{2\cdot 2\sin^2\frac{x}{2}}=\frac{2\sin\frac{n+1}{2}x\cdot\cos\frac{n-1}{2}x-2\sin\frac{n+1}{2}x\cdot\cos\frac{n+1}{2}x}{4\sin^2\frac{x}{2}}=$$

$$\frac{2\sin\frac{n+1}{2}x\left(\cos\frac{n-1}{2}x-\cos\frac{n+1}{2}x\right)}{4\sin^2\frac{x}{2}}$$

ga ega bo'lamiz.

$$\cos\alpha - \cos\beta = -2\sin\frac{\alpha+\beta}{2}\cdot\sin\frac{\alpha-\beta}{2}$$

formulaga binoan

$$\cos\frac{n-1}{2}x - \cos\frac{n+1}{2}x = 2\sin\frac{n}{2}x\sin\frac{x}{2}$$

bo'lgani uchun

$$A = \frac{\sin\frac{n+1}{2}x\cdot\sin\frac{n}{2}x\sin\frac{x}{2}}{\sin^2\frac{x}{2}}$$

yoki

$$A = \frac{\sin\frac{n}{2}x\sin\frac{n+1}{2}x}{\sin\frac{x}{2}}$$

hosil bo'ladi.

Shunday qilib

$$\sin x + \sin 2x + \sin 3x + \ldots + \sin nx = \frac{\sin\frac{n}{2}x\sin\frac{n+1}{2}x}{\sin\frac{x}{2}}$$

formulani isbotladik.

Agar kompleks sonning trigonometrik shakli $z=r(\cos\varphi+i\sin\varphi)$ dagi qavs ichidagi ifodani Eyler formulasidan foydalanib $\cos\varphi+i\sin\varphi=e^{i\varphi}$ ko'rinishda tasvirlasak kompleks son

$$z=r\,e^{i\varphi} \qquad (30.11)$$

ko'rinishni oladi. $re^{i\varphi}$ ifoda z kompleks sonning **ko'rsatkichli shakli** deb ataladi.

2-misol. 1) $z_1=-1$, 2) $z_2=2i$, 3) $z_3=\sqrt{3}+i$, 4) $z_4=-1$ sonlar ko'rsatkichli shaklda ifodalansin.

Yechish. 1) $z_1=-1$ bo'lganda $r=|-1|=1$, $\varphi=\pi$ bo'lib $-1=\cos\pi+i\sin\pi=e^{\pi i}$ bo'ladi.

2) $z_2=2i$ bo‘lganda $r=|2i|=2$, $\varphi=\frac{\pi}{2}$ bo‘lib

$2i=2\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)=2e^{\frac{\pi}{2}i}$ bo‘ladi.

3) $z_3=\sqrt{3}+I$ bo‘lganda $r=|z_3|=\sqrt{(\sqrt{3})^2+1^2}=2$, $\varphi=\mathrm{arctg}\frac{1}{\sqrt{3}}=\frac{\pi}{6}$

bo‘lganligi sababli $\sqrt{3}+i=2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)=2e^{\frac{\pi}{6}i}$ bo‘ladi.

4) $z_4=-i$ uchun $r=|-i|=1$, $\varphi=\frac{3\pi}{2}$ bo‘lganligi sababli

$-i=\left(\cos\frac{3\pi}{2}+i\sin\frac{3\pi}{2}\right)=e^{\frac{3\pi}{2}i}$ bo‘ladi.

Ko‘rsatkichli funksiyaning xossalariga asosan ko‘rsatkichli shakldagi kompleks sonlar ustida ko‘paytirish, bo‘lish, darajaga ko‘tarish va ildiz chiqarish amallarini bajarish oson.

$z_1=r_1e^{i\varphi_1}$, $z_2=r_2e^{i\varphi_2}$ kompleks sonlar berilgan bo‘lsin.

(30.2) tenglikka binoan

$$z_1\cdot z_2=r_1e^{i\varphi_1}\cdot r_2e^{i\varphi_2}=r_1\cdot r_2e^{i(\varphi_1+\varphi_1)}$$

bo‘ladi. Bu trigonometrik shaklda berilgan kompleks sonlarni ko‘-paytmasini topish uchun chiqarilgan (30.4) formula bilan bir xil.

(30.5) formulaga muvofiq

$$\frac{z_1}{z_2}=\frac{r_1e^{i\varphi_1}}{r_2e^{i\varphi_2}}=\frac{r_1}{r_2}\cdot e^{i(\varphi_1-\varphi_2)}.$$

Bu trigonometrik shaklda berilgan kompleks sonlarni bo‘lin-masini topish uchun chiqarilgan (30.5.) formula bilan bir xil.

$$z^n=(re^{i\varphi})^n=r^ne^{in\varphi}$$

Bu Muavr formulasidir.

$$\sqrt[n]{re^{i\varphi}}=\sqrt[n]{r}\cdot e^{\frac{\varphi+2k\pi}{n}i} \qquad (k=0,1,2,...n-1)$$

formula kompleks sondan ildiz chiqarish formulasi (30.8) bilan bir xil.

Endi kompleks son tushunchasining kiritilish sababi mavzusida qo'yilgan masalaga qaytamiz, ya'ni nima uchun

$$\frac{1}{1+x^2}=1-x^2+x^4-x^6+\ldots. \qquad (\delta)$$

tenglik faqatgina *-1<x<1* bo'lganda o'rinli. Bu savolga x haqiqiy sonlar sohasida bo'lganda javob beraolmaymiz, chunki $\frac{1}{1+x^2}$ funksiya $(-\infty,+\infty)$ oraliqda aniqlangan va uzluksiz. Jumladan x ning *-1* va *1* qiymatlarida ham funksiya uzluksiz. Bundan $S=1-x^2+x^4-x^6+\ldots$ yig'indi $x\leq-1$ va $x\geq1$ bo'lganda $\frac{1}{1+x^2}$ funksiyani ifodalamasligiga sabab topaolmaymiz.

Agar $S=1-x^2+x^4-x^6+\ldots$ kompleks sonlar sohasida qaralsa masala oydinlashadi. $\frac{1}{1+x^2}$ kasrning maxraji $x=\pm i$ qiymatlarda nolga aylanadi, natijada erkli o'zgaruvchining bu qiymatlari qaralayotgan funksiyaning maxsus (uzilish) nuqtalari bo'ladi. Agar kompleks sonni ⓩ tekislikning nuqtasi ko'rinishda tasvirlasak $x=\pm i$ maxsus nuqtalar markazi koordinatalar boshida bo'lib radiusi *1* teng aylanada yotadi.

Shunday qilib $\frac{1}{1+x^2}$ funksiya ⓩ tekislikdagi markazi koordinatalar boshida bo'lib radiusi *1* ga teng doiraning ichida maxsus nuqtaga ega bo'lmaydi, ammo *u* bu doiraning aylanasida ikkita maxsus nuqtalarga ega.

Ana shu $|x|\geq1$ bo'lganda (δ) tenglikni bajarilmasligiga sabab bo'ladi.

O'z-o'zini tekshirish uchun savollar

1. Kompleks o'zgaruvchining kompleks funksiyasini ta'riflang.

2. Kompleks o'zgaruvchining ko'rsatkichli funksiyasini ta'riflang va xossalarini ayting.

3. Haqiqiy o'zgaruvchining kompleks funksiyasini limitini ta'riflang.

4. Haqiqiy o'zgaruvchining kompleks funksiyasini haqiqiy o'zgaruvchi bo'yicha hosilasini ta'riflang.

5. Ko'rsatkichli funksiyaning hosilasini topish formalasini keltirib chiqaring.

6. Eyler formulasini yozing.

7. Kompleks sonning ko'rsatkichli shaklini yozing.

8. Kompleks sonning ko'rsatkichli shakli yordamida kompleks sonlarni ko'paytirish, bo'lish, darajaga ko'tarish va ildiz chiqarish qoidalarini chiqaring.

Mustaqil yechish uchun mashqlar

Kompleks sonlar ko'rsatkichli shaklda yozilsin:

1. $1+i$. Javob: $\sqrt{2}e^{\frac{\pi}{4}i}$. 2. $1-i$. Javob: $\sqrt{2}e^{\frac{7\pi}{2}i}$.

3. i. Javob: $e^{\frac{\pi}{2}i}$. 4. -2. Javob: $2e^{\pi i}$.

5. $-4i$. Javob: $4e^{\frac{3\pi}{2}i}$. 6. 3. Javob: $3e^{0i}$.

Hosilalar topilsin

7. $w=x^2+i(3+e^x)$. Javob: $w'_x=2x+ie^x$.

8. $w=\ell n(1+x^2)+i\sin x^3$. Javob: $w'_x=\dfrac{2x}{1+x^2}+i3x^2\cos x^3$.

9. $w=3^x+i\operatorname{arctg}x^2$. Javob: $w'_x=3^x\ell n3+i\dfrac{2x}{1+x^4}$.

10. $w=\cos^2 x+i\sqrt{1+x^2}$. Javob: $w'_x=-\sin 2x+i\dfrac{x}{\sqrt{1+x^2}}$.

11. $\cos x+\cos 2x+\cos 3x+...+\cos nx$ yig'indi topilsin Javob: $\dfrac{\sin\frac{nx}{2}\cos\frac{n+1}{2}x}{\sin\frac{x}{2}}$.

3_1. KO'PHADLAR

31.1. Kompleks sohada ko'phadlar. Ko'phadning ildizi. Bezu teoremasi

Ta'rif. *n* natural son bo'lganda

$$F(x)=a_0x^n+a_1x^{n-1}+...+a_n$$

ko'rinishdagi funksiya **ko'phad** yoki **butun ratsional funksiya** deb ataladi; *n* soni ko'phadning **darajasi** deyiladi. Bu yerda $a_0\neq 0$, $a_1,..., a_n$ koeffitsientlar ma'lum haqiqiy yoki kompleks sonlar; erkli o'zgaruvchi *x* ham haqiqiy, ham kompleks qiymatlar olishi mumkin. O'zgaruvchi *x* ning ko'phadni nolga aylantiradigan qiymati ko'phadning **ildizi** deb ataladi.

Birinchi darajali $y=ax+b$ ko'phad **chiziqli funksiya** deb ataladi; ikkinchi darajali $y=ax^2+bx+c$ $(a\neq 0)$ ko'phad **kvadrat uchhad** deb ataladi. $y=a_0$ o'zgarmas sonni nolinchi darajali ko'phad sifatida qabul qilamiz.

Ammo ko'phad deyilganda biz darajasi noldan farqli ko'phadlarni nazarda tutamiz.

Ko'phadni *n*-darajali ko'phad ekanligini ta'kidlash maqsadida uni ba'zan $P_n(x)$ kabi yoziladi.

Ikkita $P(x)$ va $Q(x)$ ko'phadlarning bir xil darajali *x* lari oldidagi barcha koeffitsientlari teng bo'lsa ular **teng** deyiladi.

Teng ko'phadlar *x* ning barcha qiymatlarida bir xil qiymatlar qabul qiladi.

Ko'phadlarni qo'shish, ayirish, ko'paytirish va bo'lish mumkin. Ko'phadlarni qo'shish, ayirish va ko'paytirish amallari maktab kursidan ma'lum usullar asosida bajariladi.

Ko'phadlarni bo'lish qoldiqsiz (butun) va qoldiqli bo'lishi mumkin. $P_n(x)$ va $D_m(x)$ ko'phadlar berilgan bo'lsin, bunda $n\geq m$. Agar $Q_{n-m}(x)$ ko'phad mavjud bo'lib

$$P_n(x)=D_m(x)\cdot Q_{n-m}(x)$$

tenglik bajarilsa $P_n(x)$ ko'phad $D_m(x)$ ko'phadga **qoldiqsiz** bo'linadi deb ataladi. Bunda $P_n(x)$ bo'linuvchi, $D_m(x)$ bo'luvchi, $Q_{n-m}(x)$ bo'linma ko'phad deyiladi.

Masalan x^2-1 ko'phad $x+1$ ko'phadga qoldiqsiz bo'linadi, chunki $x-1$ ko'phad mavjud bo'lib $x^2-1=(x+1)(x-1)$ tenglik bajariladi.

Ko'phadlarni qoldiqsiz bo'lish har doim ham bajarilavermaydi, masalan x^2+1 ko'phad $x-1$ ko'phadga qoldiqsiz bo'linmaydi.

Agar $Q_{n-m}(x)$ ko'phad hamda darajasi $D_m(x)$ ko'phadning darajasi m dan kichik bo'lgan $R(x)$ ko'phadlar mavjud bo'lib

$$P_n(x)=D_m(x)\cdot Q_{n-m}(x)+R(x)$$

tenglik bajarilsa $P_n(x)$ ko'phad $D_m(x)$ ko'phadga **qoldiqli** bo'linadi deb ataladi. Bunda $P_n(x)$ bo'linuvchi, $D_m(x)$ bo'luvchi, $Q_{n-m}(x)$ bo'linma, $R(x)$ – qoldiq ko'phadlardir.

Ko'phadlarni qoldiqli bo'lish har doim ham bajariladi.

Ko'phadlarni bo'lish amali bo'luvchining darajasi bo'linuvchining darajasidan katta bo'lmagandagina bajarilishini yana bir bor ta'kidlaymiz.

Ko'phadlarni bo'lishdan chiqadigan bo'linma va qoldiqni topishning har xil usullari mavjud. Ularni sonlarni bo'lishda foydalaniladigan «burchakli bo'lish» qoidasidan foydalanib topish ham mumkin.

1-misol. $P_4(x)$ $3x^4-2x^3+x^2+5x+1$ ko'phadni $D_3(x)=$ x^3+2x^2+3x+4 ko'phadga bo'ling. Bo'linma va qoldiqni toping.

Yechish.

$$\begin{array}{l|l} 3x^4-2x^3+x^2+5x+1 & x^3+2x^2+3x+4 \\ \cline{2-2} 3x^4+6x^3+9x^2+12x & 3x-8 \\ \cline{1-1} -8x^3-8x^2-7x+1 & \\ -8x^3-16x^2-24x-32 & \\ \cline{1-1} 8x^2+17x+33\,. & \end{array}$$

Bundan $Q_1(x)=3x-8$-bo‘linma, $R(x)=8x^2+17x+33$ qoldiq hadga ega bo‘lamiz.

Demak,

$3x^4-2x^3+x^2+5x+1=(x^3+2x^2+3x+4)(3x-8)+(8x^2+17x+33).$

2-мисол. $P_4(x)=x^4-1$ ko‘phad $D_1(x)=x+1$ ko‘phadga bo‘linsin. Bo‘linma va qoldiq topilsin.

Yechish. Qisqa ko‘paytirish formulasi a^2-$в^2$=(a+в)(a-в) ga binoan

$$P_4(x)=(x^2)^2-1^2=(x^2-1)(x^2+1)=(x-1)(x+1)(x^2+1).$$

Demak, $Q_3(x)=(x-1)(x^2+1)=x^3-x^2+x-1$ bo‘linma, qoldiq $R(x)=0$. Ya‘ni x^4-1 ko‘phad $x+1$ ko‘phadga qoldiqsiz bo‘linar ekan.

$P_n(a)=0$ bo‘lsa $x=a$ son $P_n(x)$ ko‘phadning ildizi bo‘lishi yuqorida ta‘kidlandi.

Endi $P_n(x)$ ko‘phadni $x-a$ ga bo‘lishdan chiqadigan qoldiqni bo‘lish jarayonini bajarmasdan topish imkonini beradigan teoremani keltiramiz.

Bezu teoremasi. $P_n(x)$ ko‘phadni $x-a$ ikki hadga bo‘lganda $P_n(a)$ ga teng qoldiq hosil bo‘ladi.

Isboti. $P_n(x)$ ko‘phadni $x-a$ ikkihadga bo‘lib

$$P_n(x)=(x-a)\,Q_{n-1}(x)+R$$

tenglikni hosil qilamiz. Bunga $x=a$ qiymatni qo‘yib isbotlanishi lozim bo‘lgan $P_n(a)=R$ tenglikka ega bo‘lamiz.

3-misol. $P_4(x)=x^4-2x^2-3$ ko‘phadni: *a)x-2; b)x-i* ikkihadlarga bo‘lishdan chiqqan qoldiqni toping.

Yechish. a) $R=P_4(2)=2^4-2\cdot2^2-3=5.$

b) $R=P_4(i)=i^4-2\cdot i^2-3=0.$

Bezu teoremasining natijasi. Agar a son $P_n(x)$ ko‘phadning ildizi ya‘ni $P_n(a)=0$ bo‘lsa, $P_n(x)$ ko‘phad $x-a$ ga qoldiqsiz bo‘linadi. Demak u

$$P_n(x)=(x-a)\,Q_{n-1}(x)$$

ko‘paytma ko‘rinishda tasvirlanadi.

Demak, $P_n(x)$ ko‘phadni $x-a$ ga qoldiqsiz bo‘linishi uchun a son $P_n(x)$ ko‘phadning ildizi bo‘lishi talab qilinar ekan.

4-misol. $P_3(x)=x^3-9x^2+26x-24$ ko‘phadni ildizlari topilsin.

Yechish. $P_3(2)=2^3-9\cdot2^2+26\cdot2-24=0$ bo‘lganligi sababli berilgan ko‘phad $x-2$ ga qoldiqsiz bo‘linadi, ya‘ni $x=2$ shu ko‘phadning ildizi bo‘ladi. Ko‘phadning boshqa ildizlarini topish uchun uni $x-2$ ga bo‘lishdan hosil bo‘ladigan bo‘linmani topamiz:

$$\begin{array}{r|l} x^3-9x^2+26x-24 & x-2 \\ \cline{2-2} x^3-2x^2 \qquad\qquad & x^2-7x+12 \\ \hline -7x^2+26x-24 & \\ -7x^2+14x \qquad & \\ \hline 12x-24 & \\ 12x-24 & \\ \hline 0 & \end{array}$$

Shunday qilib, $P_3(x)=(x-2)(x^2-7x+12)$ tenglikka ega bo‘lamiz. Agar $x^2-7x+12=0$ kvadrat tenglamaning ildizlarini topsak $P_3(x)$ ko‘phadning qolgan ildizlarini topgan bo‘lamiz:

$$x_{1,2}=\frac{7\pm\sqrt{49-4\cdot1\cdot12}}{2}=\frac{7\pm1}{2},\qquad x_1=3,\quad x_2=4.$$

Demak, $P_3(x)=(x-2)(x-3)(x-4)$ va $2,3,4$ sonlar berilgan ko‘phadning ildizlari bo‘lar ekan.

31.2. Algebraning asosiy teoremasi. Ko‘phadni chiziqli ko‘paytuvchilarga ajratish

Ko‘phadni darajasi noldan farqli bir nechta ko‘phadlarning ko‘paytmasi shaklida tasvirlash uni **ko‘paytuvchilarga ajratish** deyiladi. Masalan, 4-misolda keltirilgan $P_3(x)$ ko‘phad uchta chiziqli ko‘paytuvchilatga ajralar ekan.

$P_n(x)$ n-darajali ($n\in N$) ko‘phad bo‘lganda $P_n(x)=0$ tenglama n-darajali algebraik tenglama deyiladi. Ta‘rifdan algebraik tenglamaning ildizlari $P_n(x)$ ko‘phadning ildizlaridan iborat ekanligi kelib chiqadi.

Birinchi darajali $ax+b=0$ $(a\neq0)$ algebraik tenglama har doim bitta $x=-\dfrac{b}{a}$ ildizga $ax^2+bx+c=0$ kvadrat tenglama

$$x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

ildizlarga ega ekanligini bilamiz. Bu holda algebraik tenglama (ko'phad)ning ildizlari uning koeffitsientlari ustida ma'lum amallarni bajarish natijasida topiladi. Uchinchi va to'rtinchi darajali algebraik tenglama (ko'phad) larning ildizlarini topish uchun ham formulalar yaratilgan. Ammo darajasi besh va undan katta bo'lgan algebraik tenglama (ko'phad) larni ildizlarini topish uchun bunaqa formulalar yaratib bo'lmasligi ko'rsatilgan.

Bu yerda biz algebraik tenglama (ko'phad)ning ildizlarini topish bilan shug'ulanmaymiz.

Agar $P_n(x) = 0$ n-darajali algebraik tenglamani haqiqiy sonlar sohasida ya'ni $P_n(x)$ ko'phadning koeffitsientlari va x haqiqiy sonlar bo'lganda qarasak, u holda bu tenglama birorta ham haqiqiy ildizga ega bo'lmasligi ham mumkin yoki n tagacha har xil haqiqiy ildizlarga ega bo'lishi ham mumkin. Masalan $x^2 + 1 = 0$ tenglama haqiqiy ildizga ega emas, $x^2 - 7x + 12 = 0$ tenglama ikkita har xil $x_1 = 3$ va $x_2 = 4$ haqiqiy ildizlarga ega, $x^2 - 2x + 1 = 0$ tenglama esa bitta $x=1$ haqiqiy ildizga ega. $P_n(x) = 0$ tenglamani kompleks sonlar sohasida qaraymiz. Ya'ni $P_n(x)$ ko'phadning koeffitsiyetlari haqiqiy yoki kompleks sonlardan iborat bo'lib x erkli o'zgaruvchi ham haqiqiy yoki kompleks qiymatlarni qabul qilsin.

Istalgan algebraik tenglama (ko'phad) ildizga egami, n-darajali algebraik tenglama (ko'phad) nechta ildizga ega bo'lishi mumkin degan savollarga javob izlaymiz.

Algebraning asosiy teoremasi. Har qanday ko'phad kamida bitta (haqiqiy yoki kompleks) ildizga ega.

Bu teoremaning isbotini hozircha keltirmaymiz. Teoremaga ko'ra har qanday algebraik tenglama kamida bitta ildizga ega ekanligi kelib chiqadi. Teoremadan foydalanib ushbu teoremani isbotlaymiz.

31.1-teorema. Har qanday n-darajali $P_n(x)=a_0x^n + a_1x^{n-1} + \ldots + a_n$

ko'phad x-α ko'rinishdagi n ta chiziqli ko'paytuvchilarga ajraladi, ya'ni:

$$P_n(x)=a_0(x-\alpha_1)(x-\alpha_2)...(x-\alpha_n) \qquad (31.1)$$

ko'rinishda tasvirlash mumkin, bu yerdagi $\alpha_1, \alpha_2... \alpha_n$ haqiqiy yoki kompleks sonlar.

Isboti. Algebraning asosiy teoremasiga binoan $P_n(x)$ ko'phad kamida bitta α_1 ildizga ega. U holda Bezu teoremasining natijasiga ko'ra bu ko'phadlarni $P_n(x)=(x-\alpha_1)Q_{n-1}(x)$ ko'rinishda tasvirlash mumkin.

Algebraning asosiy teoremasiga binoan $Q_{n-1}(x)$ ko'phad ham kamida bitta α_2 ildizga ega. U holda Bezu teoremasining natijasiga ko'ra ko'phadni

$$Q_{n-1}(x)=(x-\alpha_2)Q_{n-2}(x)$$

ko'rinishda tasvirlash mumkin.

Shu jarayonni davom ettirib,

$$Q_1(x)=(x-\alpha_n)Q_0(x)$$

tenglikka ega bo'lamiz, bunda Q_0 biror son. Bu son x^n oldidagi koeffitsient a_0 ga teng bo'lishi ravshan,ya'ni $Q_0=a_0$.

Topilgan tengliklarga asoslanib isbotlanishi lozim bo'lgan (31.1) munosabatni hosil qilamiz. (31.1) tenglikdan $\alpha_1, \alpha_2... \alpha_n$ sonlar $P_n(x)$ ko'phadning ildizlari ekanligi kelib chiqadi, chunki

$$P_n(\alpha_1)=P_n(\alpha_2)=...=P_n(\alpha_n)=0.$$

Xulosa: n-darajali ko'phad n tadan ortiq har xil ildizlarga ega bo'laolmaydi.

31.3. Haqiqiy koeffitsientli ko'phadni chiziqli va kvadrat uchhad ko'rinishdagi ko'paytuvchilarga ajratish

Agar n-darajali ko'phadning chiziqli ko'paytuvchilarga yoyilmasi

$$P_n(x)=a_0(x-\alpha_1)(x-\alpha_2)...(x-\alpha_n)$$

da ba'zi chiziqli ko'paytuvchilar bir xil bo'lsa, ularni birlashtirish mumkin. U holda (31.1) yoyilma

$$P_n(x)=a_0(x-\alpha_1)^{\kappa_1}(x-\alpha_2)^{\kappa_2}...(x-\alpha_n)^{\kappa m}$$

ko'rinishga ega bo'ladi, bunda $\kappa_1+\kappa_2+...+\kappa_m=n$. Bu holda α_1 ildizga ko'phadning κ_1 **karrali** ildizi, α_2 uning κ_2 **karrali** ildizi deyiladi va hokazo. Ko'phadning 1 karrali ildizi uning **oddiy** ildizi deyiladi.

Masalan, $P_6(x)=(x+1)^3(x-2)^2(x+3)$ ko'phad uchun $x=-1$ uch karrali, $x=2$ ikki karrali, $x=-3$ esa oddiy ildiz bo'ladi.

Agar ko'phad κ karrali ildizga ega bo'lsa u κ ta bir xil ildizlarga ega deb hisoblanadi.

Xulosa: Har qanday n-darajali ko'phad roppa-rosa n ta ildizga (haqiqiy va kompleks) ega.

Demak, kompleks sonlar sohasidagi har qanday n-darajali algebraik tenglama roppa-rosa n ta ildizlarga ega ekan. Algebraik bo'lmagan tenglamalar ildizga ega bo'lmasligi ham mumkin. Masalan $e^z=0$ tenglama ildizga emas.

Haqiqatan. $e^z=e^{x+iy}=e^x(\cos y+i\sin y)$ tenglikda $e^x>0$ va $\cos y+i\sin y\neq 0$, chunki $\cos y+i\sin y=0$ tenglik faqatgina $\cos y=0, \sin y=0$ bo'lganda bajariladi. $\cos^2 x+\sin^2 x=1$ bo'lgani uchun bir vaqtda $\cos y=0, \sin y=0$ bajarilmaydi. Demak z ning hech bir qiymatida $e^z=0$ tenglik bajarilmaydi, ya'ni $e^z=0$ tenglama kompleks sonlar sohasida ham ildizga ega emas.

31.2-teorema. Agar $\alpha=\gamma+i\delta$ kompleks son haqiqiy koeffitsientli $P_n(x)$ ko'phadning ildizi bo'lsa, u holda unga qo'shma $\bar{\alpha}=\gamma-i\delta$ son ham shu ko'phadning ildizi bo'ladi.

Bu teoremani isbotsiz qabul qilamiz.

Bu teoremaga ko'ra (31.1) yoyilmada kompleks ildizlar o'z qo'shma juftlari bilan qatnashadilar.

(31.1) yoyilmadagi kompleks qo'shma ildizlarga mos keluvchi chiziqli ko'paytuvchilarni ko'paytiramiz:

$$(x-\alpha)(x-\bar{\alpha})=[x-(\gamma+i\delta)][x-(\gamma-i\delta)]=[(x-\gamma)-i\delta][(x-\gamma)+i\delta]=$$
$$=(x-\gamma)^2-(i\delta)^2=x^2-2x\gamma+\gamma^2+\delta^2,$$

$-2\gamma=p, \gamma^2+\delta^2=q$ deb belgilasak

$$(x-\bar{a})(x-\bar{a})=x^2+px+q$$

tenglikka ega bo'lamiz, bunda p va q - haqiqiy sonlar.

Shunday qilib yoyilmadagi qo'shma kompleks ildizlarga mos keladigan chiziqli ko'paytuvchilar ko'paytmasini haqiqiy koeffitsientli kvadrat uchhad bilan almashtirish mumkin ekan.

Agar $\alpha=\gamma+i\delta$ kompleks son ko'phadning k karrali ildizi bo'lsa, u holda $\overline{\alpha}=\gamma-i\delta$ qo'shma kompleks son ham shu ko'phadning k karrali ildizi bo'ladi.

Bu holda $(x-\alpha)^k(x-\overline{\alpha})^k$ ko'paytmani $(x^2+pk+q)^k$ bilan almashtirish mumkin.

Shunday qilib, **har qanday haqiqiy koeffitsientli *n*-darajali ko'phad darajalari natural sonlardan iborat chiziqli va haqiqiy koeffitsientli kvadrat uchhad shaklidagi ko'paytuvchilarga ajraladi,** ya'ni:

$$P_n(x)=a_0(x-\alpha_1)^{\kappa_1}\cdot(x-\alpha_2)^{\kappa_2}\ldots(x-\alpha_r)^{\kappa_r}\cdot(x^2+p_1x+q_1)^{s_1}\text{x}$$

$$\text{x}(x^2+p_2x+q_2)^{s_2}\ldots(x^2+p_ex+q_e)^{s_e},$$

bunda $\kappa_1+\kappa_2+\ldots+\kappa_r+2s_1+2s_2+\ldots+2s_e=n$ va $\alpha_1, \alpha_2,\ldots \alpha_r, p_1, q_1,\ldots p_e, q_e$ haqiqiy sonlar.

5-misol. $P_8(x)=x^8-x^6-x^2+1$ ko'phadni ko'paytuvchilarga ajrating.

Yechish. $P_8(x)=x^8-x^2-x^6+1=x^2(x^6-1)-(x^6-1)=(x^2-1)(x^6-1)=$
$=(x-1)(x+1)((x^3)^2-1^2)=(x-1)(x+1)(x^3-1)(x^3+1)=(x-1)(x+1)(x-1)$x
x$(x^2+x+1)(x+1)(x^2-x+1)=(x-1)^2(x+1)^2(x^2+x+1)(x^2-x+1)$.

O'z-o'zini tekshirish uchun savollar

1. Ko'phad deb nimaga aytiladi?
2. Ko'phadning ildizi nima?
3. Qachon ko'phadlar teng deyiladi?
4. Ko'phadlarni qo'shish, ayirish, ko'paytirish qoidalari qanday?
5. Ko'phadlarni qoldiqsiz bo'lish deb nimaga aytiladi?
6. Ko'phadlarni qoldiqli bo'lish deb nimaga aytiladi?
7. Bezu teoremasini isbotlang.
8. Bezu teoremasining natijasi nimadan iborat?
9. Algebraning asosiy teoremasini bayon eting.
10. Ko'phadni chiziqli ko'paytuvchilarga ajratish haqidagi teoremani bayon eting.
11. Ko'phadning karrali va oddiy ildizlarini ta'riflang.

12. n-darajali ko‘phad nechta ildizga ega?

13. Haqiqiy koeffitsientli ko‘phadni ko‘paytuvchilarga ajratish jarayonini tushuntirib bering.

Mustaqil yechish uchun mashqlar

1. $P_3(x)= x^3-4x^2+8x-1$ ko‘phad $x+4$ ga bo‘linsin.
Javob: $P_3(x)=(x+4)(x^2-8x+40)-161$, ya‘ni bo‘linma $Q_2(x)= x^2-8x+40$, qoldiq
$R= P_3(-4)=-161$.

2. $P_4(x)=x^4+12x^3+54x^2+108x+81$ ko‘phad $x+3$ ga bo‘linsin.
Javob: $P_4(x)=(x+3)(x^3+9x^2+27x+27)$, ya‘ni bo‘linma
$Q_3(x)= x^3+9x^2+27x+27$, qoldiq $R= P_4(-3)=0$.

3. $P_7(x)=x^7-1$ ko‘phad $x-1$ ga bo‘linsin.
Javob: $P_7(x)=(x-1)(x^6+x^5+x^4+x^3+x^2+x+1)$, ya‘ni ko‘phad $x-1$ ga qoldiqsiz bo‘linadi, chunki $P_7(1)=1^7-1=0$.

Ushbu haqiqiy koeffitsientli ko‘phadlar ko‘paytuvchilarga ajratilsin

4. $P_4(x)=x^4-1$. Javob: $P_4(x)=(x-1)(x+1)(x^2+1)$.

5. $P_4(x)=x^4+1$. Javob: $P_4(x)=(x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)$.

6. $P_5(x)= x^5+2x^4-x-2$. Javob: $P_5(x)=(x-1)(x+1)(x+2)(x^2+1)$.

Ushbu ko‘phadlarning ildizlari topilsin

7. $P_2(x)=x^2+1$. Javob: $i, -i$.

8. $P_3(x)=x^3+x^2+4x+4$. Javob: $2i, -2i, -1$.

9. $P_3(x)=x^3+1$. Javob: $-1,\ \frac{1}{2}+\frac{\sqrt{3}}{2}i,\ \frac{1}{2}-\frac{\sqrt{3}}{2}i$.

32. BOSHLANG‘ICH FUNKSIYA VA ANIQMAS INTEGRAL TUSHUNCHALARI

32.1. Boshlang‘ich funksiya tushunchasi

Hosila tushunchasining kiritilishiga sabab bo‘lgan masalalardan biri to‘g‘ri chiziqli harakatda moddiy nuqtaning oniy tezligi $V(t)$ ni topish masalasi edi. Ya’ni $S=f(t)$ (t-vaqt, S-yo‘l) harakat qonuni asosida to‘g‘ri chiziq bo‘ylab bir tomonlama harakat qilayotgan moddiy nuqtaning oniy tezligi $V(t)=S'(t)=f'(t)$ kabi topilar edi. Bu yerda berilgan $f(t)$ funksiyaga ko‘ra uning $V(t)$ hosilani topish talab etilar edi. Endi teskari masalani qaraymiz, ya’ni yuqorida eslatilgan masalada t $(t_0 \le t \le T)$ vaqtning istalgan paytida oniy tezlik $V(t)$ ma’lum bo‘lganda shu paytgacha o‘tilgan $S(t)$ yo‘lni topish talab etilsin, ya’ni $S(t)$ funksiyaning ma’lum hosilasi $S'(t)=V(t)$ ga ko‘ra uning o‘zini topish talab etilsin.

Bu va bu kabi hayotiy masalalarni fizik mohiyatini e’tiborga olmasdan matematik nuqtai nazardan o‘rganishga kirishamiz, ya’ni $F(x)$ funksiyani uning ma’lum $F'(x)=f(x)$ hosilasiga yoki $f(x)dx$ differensialiga ko‘ra topish amali bilan shug‘ullanamiz.

1-ta‘rif. Biror oraliqda aniqlangan $f(x)$ funksiya uchun shu oraliqning barcha nuqtalarida $F'(x)=f(x)$ yoki $dF(x)=f(x)dx$ shart bajarilsa, u holda $F(x)$ funksiya shu oraliqda $f(x)$ ning **boshlang‘ich funksiyasi** deyiladi.

Masalan, $F(x)=\sin x$ funksiya butun son o‘qida $f(x)=\cos x$ funksiyaning boshlang‘ich funksiyasi bo‘ladi, chunki istalgan x uchun

$F'(x)=(\sin x)'=\cos x=f(x)$. Shuningdek $F(x)=\sqrt{1-x^2}$ funksiya (-1,1) intervalda $f(x)=-\dfrac{x}{\sqrt{1-x^2}}$ funksiyaning boshlang‘ich funksiyasi bo‘ladi, chunki intervaldan olingan barcha x lar uchun

$$F'(x)=(\sqrt{1-x^2})'=-\frac{x}{\sqrt{1-x^2}}=f(x).$$

Ixtiyoriy o'zgarmas C uchun $F(x)=\frac{x^3}{3}+C$ funksiya butun son o'qida $f(x)=x^2$ funksiyaning boshlang'ich funksiyasi bo'ladi, chunki istalgan x uchun $F'(x)=\left(\frac{x^3}{3}+C\right)'=x^2$. Oxirgi misol funksiyaning boshlang'ich funksiyasi yagona bo'lmasligini ko'rsatadi.

1-eslatma. $f(x)$ funksiyaning boshlang'ich funksiyasi $F(x)$ (agar u mavjud bo'lsa) uzluksiz funksiya bo'ladi.

Haqiqatdan boshlang'ich funksiyaning ta'rifiga binoan $F'(x)$ hosila mavjud va $F'(x)=f(x)$. Differensiallanuvchi funksiyaning uzluksizligidan $F(x)$ ning uzluksizligi kelib chiqadi.

Endi $F(x)$ funksiya $f(x)$ ning istalgan boshlang'ich funksiyasi bo'lganda uning qolgan barcha boshlang'ich funksiyalari $F(x)+C$ ko'rinishga ega bo'lishni ko'rsatamiz.

Bundan keyin C orqali ixtiyoriy o'zgarmas son belgilanadi.

32.1-lemma. Biror oraliqda hosilasi nolga teng $f(x)$ funksiya shu oraliqda o'zgarmasdir.

Isboti. Shartga ko'ra oraliqdagi barcha x uchun $f'(x)=0$. Oraliqqa tegishli $x_1<x_2$ qiymatlarni olib

$$f(x_2)-f(x_1)=f'(z)(x_2-x_1), \quad x_1<z<x_2$$

Lagranj formulasini yozamiz. $f'(z)=0$ bo'lganligi uchun

$f(x_2)-f(x_1)=0$ yoki $f(x_2)=f(x_1)$ tenglikka ega bo'lamiz. Bu $f(x)$ funksiyaning qaralayotgan oraliqning istalgan nuqtalaridagi qiymatlari bir xil ekanligini ya'ni u o'zgarmasligini ko'rsatadi.

32.1-teorema. Agar $F(x)$ va $\phi(x)$ funksiyalar $f(x)$ funksiyaning biror oraliqdagi boshlang'ich funksiyalari bo'lsa, u holda ular bir-biridan o'zgarmas songa farq qiladi: $\phi(x)-F(x)=C$.

Isboti. $\phi(x)$ funksiya $f(x)$ funksiyaning qaralayotgan oraliqdagi $F(x)$ dan farqli boshqa bir boshlangich funksiyasi bo'lsin, ya'ni $\phi'(x)=f(x)$.

U holda shu oraliqdagi ixtiyoriy x uchun

$[\phi(x)-F(x)]'=\phi'(x)-F'(x)=f(x)-f(x)=0$ bo'ladi.

32.1- lemmaga muvofiq

$\phi(x)-F(x)=C$ yoki $\phi(x)=F(x)+C$ bo'ladi.

Bu teoremadan quyidagi natija kelib chiqadi.

Natija. Agar $F(x)$ funksiya $f(x)$ ning biror orliqdagi boshlang'ich funksiyasi bo'lsa, u holda uning shu oraliqdagi istalgan boshlang'ich funksiyasi $\phi(x)= =F(x)+C$ ko'rinishga ega bo'ladi.

2-eslatma. Har qanday funksiya ham boshlang'ich funksiyaga ega bo'lavermaydi.

1-misol. $x=0$ nuqtada uzilishga ega

$$f(x)=\begin{cases} -1, & agar \quad -1<x<0 \quad bo'lsa, \\ 0, & agar \quad x=0 \quad bo'lsa, \\ 1, & agar \quad 0<x<1 \quad bo'lsa \end{cases}$$

funksiyani qaraymiz.

Bu funksiya $(-1,1)$ intervalda boshlang'ich funksiyaga ega emasligini ko'rsatamiz. Teskarisini faraz qilamiz. $(-1,1)$ intervalda $f(x)$ funksiya uchun boshlang'ich funksiya $F(x)$ mavjud bo'lsin. U holda ta'rifga binoan $(-1,1)$ intervalga tegishli barcha x lar uchun $F'(x)$ hosila mavjud bo'lib $F'(x)=f(x)$ tenglik o'rinli bo'ladi. Jumladan $F'(0)=f(0)$ tenglik ham to'g'ri bo'ladi. $(0,1)$ intervalga tegishli x qiymatni olib $[0,x]$ kesmani qaraymiz. $F(x)$ funksiya $[0,x]$ kesmada uzluksiz, $(0,x)$ intervalda differensiallanuvchi bo'lganligi sababli Lagranj formulaga binoan shunday z $(0<z<x)$ qiymat mavjud bo'lib

$$F(x)-F(0)=F'(z)(x-0)=f(z)\cdot x=1\cdot x=x$$

tenglik o'rinli bo'ladi.

Bundan $\dfrac{F(x)-F(0)}{x-0}=1$ tenglikka yoki hosilaning ta'rifiga asosan $F'(0)=F'(+0)=\lim\limits_{x\to+0}\dfrac{F(x)-F(0)}{x-0}=1$ tenglikka ega bo'lamiz. Bu tenglik $F'(0)=f(0)=0$ ra zid. Bu ziddiyatga berilgan $f(x)$

funksiya uchun *F (x)* boshlang'ich funksiya mavjud deb qilgan noto'g'ri farazimiz oqibatida keldik.

2-misol. $x=0$ nuqtada uzilishga ega bo'lgan

$$f(x)=\begin{cases}+1, & \text{agar } -1<x\le 0 \text{ bo' lsa,}\\ 2x, & \text{agar } \ 0<x<1 \text{ bo' lsa}\end{cases}$$

funksiyani qaraymiz. Bu funksiya uchun

$$F(x)=\begin{cases}x, & \text{agar } -1<x\le 0 \text{ bo' lsa,}\\ x^2, & \text{agar } \ 0<x<1 \text{ bo' lsa}\end{cases}$$

funksiya $(-1;1)$ oraliqda boshlang'ich funksiya bo'lishi ko'rinib turibdi.

Bu misollardan ko'rinib turibdiki uzilishga ega funksiyalar orasida boshlang'ich funksiyaga ega bo'lganlari ham ega bo'lmaganlari ham mavjud ekan.

32.2-teorema. Biror oraliqda uzluksiz funksiya shu oraliqda borshlang'ich funksiyaga ega.

Bu teoremaning isboti keyinroq keltiriladi.

32.2. Funksiyaning aniqmas integrali

2-ta'rif. Agar $F(x)$ funksiya $f(x)$ ning boshlang'ich funksiyalaridan biri bo'lsa, u holda $F(x)+C$ ifoda $f(x)$ funksiyaning aniqmas integrali deyiladi va $\int f(x)dx$ kabi belgilanadi. Bunda $\int$ – integral belgisi, $f(x)$-integral ostidagi funksiya, $f(x)dx$ –integral ostidagi ifoda, x- integrallash o'zgaruvchisi deyiladi.

Demak, ta'rifga binoan $F'(x)=f(x)$ bo'lganda $\int f(x)dx=F(x)+C$ tenglik o'rinli bo'lar ekan.

Demak, $\int\frac{dx}{x}=\ell n|x|+C$, chunki $(\ell n|x|+C)'=\frac{1}{x}$,

$\int\cos xdx=\sin x+C$, chunki $(\sin x+C)'=\cos x$,

$\int\frac{dx}{\cos^2 x}=tgx+C$, chunki $(tgx+C)'=\frac{1}{\cos^2 x}$,

$$\int x^{\alpha}dx = \frac{x^{\alpha+1}}{\alpha+1} + C, \quad (\alpha \neq -1), \text{chunki} \left(\frac{x^{\alpha+1}}{\alpha+1} + C\right)' = x^{\alpha}.$$

f(x) funksiyaning boshlang'ich funksiyasini (aniqmas integralini) topish jarayoni shu funksiyani **integrallash** deyiladi.

Funksiyalarni differensiyallash va integrallash amallari o'zaro teskari amallardir.

Oliy matematika kursining funksiyalarni integrallash jarayoni bilan shug'ullanadigan bo'limi integral hisob deb yuritiladi.

32.3. Aniqmas integralning asosiy xossalari

1. Aniqmas integralning hosilasi integral ostidagi funksiyaga teng, ya'ni

$$\left(\int f(x)dx\right)' = f(x).$$

2. Aniqmas integralning differensiali integral ostidagi ifodaga teng, ya'ni

$$d\int f(x)dx = f(x)\,dx.$$

3. Biror funksiyaning hosilasidan olingan aniqmas integral shu funksiya bilan ixtiyoriy o'zgarmasning yig'indisiga teng, ya'ni

$$\int F'(x)\,dx = F(x) + C.$$

4. Biror funksiyaning differensialidan olingan aniqmas integral shu funksiya bilan ixtiyoriy o'zgarmasning yig'indisiga teng, ya'ni

$$\int dF(x) = F(x) + C.$$

Keltirilgan xossalardan birini, masalan 1-xossani isbotlaymiz.

Haqiqatan ham,

$$\left(\int f(x)dx\right)' = (F(x)+C)' = F'(x) + C' = F'(x) + 0 = f(x).$$

5. O'zgarmas ko'patuvchini integral belgisidan tashqarisiga chiqarish mumkin, ya'ni agar $A=const$ bo'lsa

$$\int Af(x)dx = A\int f(x)dx \tag{32.1}$$

bo'ladi.

6. Chekli sondagi funksiyalarning algebraik yigindisidan olingan aniqmas integral shu funksiyalarning har biridan olingan aniqmas integrallarning (agar ular mavjud bo'lsa) algebraik yig'indiga teng, ya'ni

$$\int [f(x) + q(x) - \varphi(x)]\,dx = \int f(x)dx + \int q(x)dx - \int \varphi(x)dx \tag{32.2}$$

Bu xossalarni to‘g‘riligini ko‘rsatish uchun tenglikning o‘ng tomonidagi ifodalarning hosilasi uning chap tomonidagi itntegral ostidagi funksiyaga tengligini ko‘rsatish kifoya.

Aniqmas integrallarning hisoblashda quyidagi qoidadan foydalanish maqsadga muvofiqdir.

Agar $\int f(x)dx = F(x)+C$ $(F'(x)) = f(x))$ bo‘lsa $\int f(ax+b)dx =$ $\frac{1}{a}F(ax+b)+C$ tenglik o‘rinli, bunda a va $в$ o‘zgarmas sonlar.

Haqiqatan. Oxirgi tenglikni o‘ng qismini differensiallasak

$$\left(\frac{1}{a}F(ax+b)+c\right)' = \frac{1}{a}F'(ax+b)\cdot(ax+b)' = \frac{1}{a}F'(ax+b)\cdot a = F'(ax+b) =$$

$= f(ax+b)$ hosil bo‘ladi. Tenglikning o‘ng tomonidagi ifodaning hosilasi uning chap tomonidagi integral ostidagi funksiyaga teng ekanligi o‘sha tenglikni to‘g‘riligini ko‘rsatadi.

Masalan,

$$\int\frac{dx}{5x-4} = \frac{1}{5}\ln|5x-4|+C, \quad \text{chunki} \quad \int\frac{dx}{x} = \ln|x|+C;$$

$$\int\cos 13x dx = \frac{1}{13}\sin 13x + C, \quad \text{chunki} \quad \int\cos x dx = \sin x + C;$$

$$\int\frac{dx}{\cos^2(6x+5)} = \frac{1}{6}tg(6x+5)+C, \quad \text{chunki} \quad \int\frac{dx}{\cos^2 x} = tgx + C.$$

32.4. Asosiy aniqmas integrallar jadvali

Elementar funksiyalarning hosilalari jadvali hamda aniqmas integralning ta‘rifidan foydalanib differensiallash yordamida bevosita tekshirib ko‘rish mumkin bo‘lgan ba‘zi-bir funksiyalarning integrallari jadvalini keltiramiz.

1. $\int 0dx = C.$ 2. $\int dx = x + C.$

3. $\int x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1}+C$, ($\alpha$-o‘zgarmas son, $\alpha \neq -1$).

4. $\int \frac{dx}{x^2} = -\frac{1}{x} + C \quad (x \neq 0).$

5. $\int \frac{dx}{\sqrt{x}} = 2\sqrt{x} + C \quad (x > 0).$ 6. $\int \frac{dx}{x} = \ln|x| + C \quad (x \neq 0).$

7. $\int e^x dx = e^x + C.$ 8. $\int a^x dx = \frac{a^x}{\ln a} + C.$

9. $\int \sin x dx = -\cos x + C.$ 10. $\int \cos x dx = \sin x + C.$

11. $\int \frac{dx}{\cos^2 x} = tgx + C, \quad (x \neq \frac{\pi}{2} + k\pi, k \in Z).$

12. $\int \frac{dx}{\sin^2 x} = -ctgx + C, \quad (x \neq \pi k, k \in Z).$

13. $\int tgx dx = -\ln|\cos x| + C, \quad (x \neq \frac{\pi}{2} + k\pi, k \in Z).$

14. $\int ctgx dx = \ln|\sin x| + C, \quad (x \neq \pi k, k \in Z).$

15. $\int \frac{dx}{1+x^2} = \begin{cases} \operatorname{arctg} x + C, \\ -\operatorname{arcctg} x + C. \end{cases}$

16. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \operatorname{arctg} \frac{x}{a} + C.$

17. $\int \frac{dx}{\sqrt{1-x^2}} = \begin{cases} \arcsin x + C, \\ -\arccos x + C, \quad (-1 < x < 1). \end{cases}$

18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \arcsin \frac{x}{a} + C. (|x| < a).$

19. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \ln\left|\frac{a+x}{a-x}\right| + C.$

20. $\frac{dx}{x^2 - a^2} = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C.$

21. $\int \frac{dx}{\sqrt{x^2 + m}} = \ln|x + \sqrt{x^2 + m}| + C.$

22. $\int \frac{xdx}{a^2 \pm x^2} = \pm \frac{1}{2} \ln\left|a^2 \pm x^2\right| + C.$

23. $\int \text{shx}dx = \text{chx} + C.$ 24. $\int \text{chx}dx = \text{shx} + C.$

25. $\int \frac{dx}{sh^2x} = -\text{cthx} + C, \quad (x \neq 0).$ 26. $\int \frac{dx}{ch^2x} = \text{thx} + C.$

Keltirilgan formulalarning o'rinli ekanligini ko'rsatish uchun tenglikning o'ng tomonidagi ifodaning hosilasi, uning chap tomondagi integral ostidagi funksiyaga teng ekanligini ko'rsatish kifoyadir.

21-formulaning to'g'riligini ko'rsatamiz.

$$(\ln(x+\sqrt{x^2+m}))' = \frac{1}{x+\sqrt{x^2+m}} \cdot (x+\sqrt{x^2+m})' = \frac{1}{x+\sqrt{x^2+m}}(1+\frac{(x^2+m)'}{2\sqrt{x^2+m}}) =$$

$$= \frac{1}{x+\sqrt{x^2+m}} \cdot \left(1+\frac{2x}{2\sqrt{x^2+m}}\right) = \frac{1}{x+\sqrt{x^2+m}} \cdot \left(\frac{\sqrt{x^2+m}+x}{\sqrt{x^2+m}}\right) = \frac{1}{\sqrt{x^2+m}}.$$

21-tenglikning o'ng tomonidagi ifodaning hosilasi uning chap tomonidagi integral ostidagi funksiyaga teng ekan, demak bu tenglik to'g'ri.

1-izoh. Keltirilgan formulalar erkli o'zgaruvchi x o'rniga uning biror funksiyasi kelganda ham o'z kuchini saqlaydi.

Masalan,

$$\int \cos x^3 dx^3 = \sin x^3 + C, \quad \int \frac{dx^2}{\sqrt{1-x^4}} = \arcsin x^2 + C, \quad \int \sin^4 x d\sin x = \frac{\sin^5 x}{5} + C$$

va hokazo.

2-izoh. Ko'paytma va bo'linmaning integrallarini topish uchun maxsus formulalar yo'q. Bu o'z navbatida funksiyani integrallash amali uni differensiallashga nisbatan ancha murakkabligidan dalolat beradi (ayrim hollarni hisobga olmaganda).

Elementar funksiyalarning hosilalari elementar funksiya bo'lishini bilamiz. Ammo hamma elementar funksiyalarning aniqmas integrallari ham elementar funksiya bo'lavermaydi. Masalan,

$$\sin x^2, \quad \cos x^2, \quad e^{x^2}, \quad \frac{\sin x}{x} \quad (x \neq 0),$$

$$\frac{\cos x}{x} \quad (x \neq 0), \quad \frac{e^x}{x} \quad (x \neq 0).$$

funksiyalar uzluksiz funksiyalar bo‘lganligi sababli ularning boshlang‘ich funksiyalari mavjud, lekin ular elementar funksiyalar emas.

O‘z-o‘zini tekshirish uchun savollar

1. Funksiyaning boshlang‘ich funksiyasi deb nimaga aytiladi?
2. Funksiya qancha boshlang‘ich funksiyalarga ega bo‘lishi mumkin?
3. Boshlang‘ich funksiyalar bir-biridan qanday farq qiladi?
4. Har qanday funksiya boshlang‘ich funksiyaga egami?
5. Qanaqa funksiyalar har doim boshlang‘ich funksiyaga ega?
6. Funksiyaning aniqmas integrali nima?
7. Aniqmas integral qanaqa xossalarga ega?
8. Asosiy aniqmas integrallar jadvalini yozing.
9. Integrallar jadvalidagi formulalarning to‘g‘riligiga qanday ishonch hosil qilinadi?
10. Boshlang‘ich funksiyasi elementar funksiya bo‘lmagan uzluksiz funksiyalarga misollar keltiring?

Mustaqil yechish uchun mashqlar

Quyidagi funksiyalar qanaqa funksiyalar uchun boshlang‘ich funksiya bo‘lishi topilsin.

1. $\frac{x^4}{4} - 4\cos x - 9x + 3.$ Javob: $x^3 + 4\sin x - 9.$

2. $\frac{2}{3}x\sqrt{x} + \frac{1}{2}\sin 2x + \ln x - 2.$ Javob: $\sqrt{x} + \cos 2x + \frac{1}{x}.$

3. $\frac{2^x}{\ln 2} - \frac{1}{2x^2} - \ln|\cos x| + 4.$ Javob: $2^x + \frac{1}{x^3} + tgx.$

4. $3tgx - \frac{2}{5}x^2\sqrt{x} - \frac{1}{x} + e.$ Javob: $\frac{3}{\cos^2 x} - x\sqrt{x} + \frac{1}{x^2}.$

5. $7\arcsin\frac{x}{2} + \frac{1}{4}\ln\left|\frac{2+x}{2-x}\right| + 6.$ Javob: $\frac{7}{\sqrt{4-x^2}} + \frac{1}{4-x^2}.$

6. $2\ln|x + \sqrt{x^2+3}| - \frac{1}{2}\ln(x^2+9) + 4.$

Javob: $\frac{2}{\sqrt{x^2+3}} - \frac{x}{x^2+9}.$

33. INTEGRALLASHNING ASOSIY USULLARI

33.1. Bevosita integrallash usuli

Bu usulda integral ostidagi funksiyani formulalar yordamida almashtirish hamda aniqmas integralning asosiy xossalari va asosiy integrallar jadvalidan foydalanib integral topiladi.

1-misol $\int(3x^2+8x-\sin x)dx$ topilsin.

Yechish. (32.2) va (32.1) formulalarga asosan

$$\int(3x^2+8x-\sin x)dx=\int 3x^2dx+\int 8xdx-\int\sin xdx=3\int x^2dx+8\int xdx-\int\sin xdx$$

tenglikka ega bo'lamiz.

Ammo integrallar jadvalidagi 3-formulaga ko'ra

$$3\int x^2dx=3\left(\frac{x^{2+1}}{2+1}+C_1\right)=3\left(\frac{x^3}{3}+C_1\right)=x^3+3C_1;$$

$$8\int xdx=8\left(\frac{x^{1+1}}{1+1}+C_2\right)=8\left(\frac{x^2}{2}+C_2\right)=4x^2+8C_2,$$

9-formulaga binoan $\int\sin xdx=-\cos x+C_3$ bo'lganligi sababli

$$\int(3x^2+8x-\sin x)dx=x^3+4x^2+\cos x+(3C_1+8C_2-C_3)$$

bo'ladi.

Har bir qo'shiluvchini integrallash natijasida o'zining o'zgarmaslari C_1, C_2 va C_3 ga ega bo'ldik. Oxirgi natijaga bitta ixtiyoriy o'zgarmas C ni yozamiz, chunki C_1, C_2, C_3 ixtiyoriy o'zgarmas bo'lganda $C=3C_1+8C_2-C_3$ ham ixtiyoriy o'zgarmas bo'ladi.

Shunday qilib

$$\int(3x^2+8x-\sin x)dx=x^3+4x^2+\cos x+C$$

tenglikka ega bo'lamiz.

Olingan natijaning to'g'riligiga differensiallash orqali ishonch hosil qilish qiyin emas.

Haqiqatan,

$$\left(x^{4}+4x^{2}+\cos x+C\right)'=3x^{3}+8x-\sin x.$$

Bundan buyon har qaysi qo'shiluvchini integrallagandan so'ng ixtiyoriy o'zgarmasni yozmaymiz, chunki bu o'zgarmasning yig'indisi yana o'zgarmas bo'lganligi uchun, uni oxirida yozamiz.

2-misol. $\int \cos^{2}\frac{x}{2}dx$ integral topilsin.

Yechish. Elementar matematikadan ma'lum $\cos^{2}\frac{x}{2}=\frac{1+\cos x}{2}$ darajani pasaytirish formulasidan foydalanib berilgan integralni $\int \cos^{2}\frac{x}{2}dx=\int\frac{(1+\cos x)dx}{2}$ ko'rinishida yozish mumkin. Oxirgi integralga (32.1) va (32.2) formulalarni qo'llasak

$\int \cos^{2}\frac{x}{2}dx=\frac{1}{2}\int(1+\cos x)dx=\frac{1}{2}\int dx+\frac{1}{2}\int\cos x dx$ kelib chiqadi. Integrallar jadvalidagi *2*- va *10*- formulalarga asoslanib

$$\int \cos^{2}\frac{x}{2}dx=\frac{1}{2}x+\frac{1}{2}\sin x+C$$

tenglikka ega bo'lamiz.

3-misol. $\int \mathrm{ctg}^{2}x dx$ integral topilsin.

Yechish. $\mathrm{ctg}^{2}x=\frac{1}{\sin^{2}x}-1$ formulaga asoslanib berilgan integralni

$$\int \mathrm{ctg}^{2}x dx=\int\left(\frac{1}{\sin^{2}x}-1\right)dx$$

ko'rinishida yozamiz. (32.2) formulaga binoan

$$\int \mathrm{ctg}^2 x dx = \int \frac{dx}{\sin^2 x} - \int dx$$

tenglikka ega bo'lamiz. Integrallar jadvalidagi *12* va *2* formulalardan foydalanib quyidagini hosil qilamiz:

$$\int \mathrm{ctg}^2 x dx = -\mathrm{ctg} x - x + C .$$

4- misol. $\int \frac{dx}{\sin^2 x \cos^2 x}$ topilsin.

Yechish.

$$\int \frac{dx}{\sin^2 x \cos^2 x} = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x} dx = \int \left(\frac{\sin^2 x}{\sin^2 x \cos^2 x} + \frac{\cos^2 x}{\sin^2 x \cos^2 x} \right) dx =$$

$$\int \left(\frac{1}{\cos^2 x} + \frac{1}{\sin^2 x} \right) dx = \int \frac{dx}{\cos^2 x} + \int \frac{dx}{\sin^2 x} = \mathrm{tg} x - \mathrm{ctg} x + C.$$

Bu yerda $\sin^2 x + \cos^2 x = 1$ trigonometrik ayniyatdan hamda aniqmas integralning xossasidan foydalanish natijasida berilgan integral jadval integrallarining yig'indisiga keltirildi.

5-misol. $\int \frac{x^3 - 2x^2 + 4x + 3}{x^2} dx$ topilsin.

Yechish.

$$\int \frac{x^3 - 2x^2 + 4x + 3}{x^2} dx =$$

$$= \int \left(\frac{x^3}{x^2} - \frac{2x^2}{x^2} + \frac{4x}{x^2} + \frac{3}{x^2} \right) dx = \int \left(x - 2 + \frac{4}{x} + \frac{3}{x^2} \right) dx =$$

$$= \int x dx - 2\int dx + 4\int \frac{dx}{x} + 3\int \frac{dx}{x^2} = \frac{x^2}{2} - 2x + 4\ln|x| - \frac{3}{x} + C.$$

Endi integral ostidagi ifodalarni ko‘rinishini o‘zgartirish hisobiga jadval integraliga keltiriladigan integrallarga misollar keltiramiz.

6-misol. $\int (x+3)^{99}dx = \int (x+3)^{99}d(x+3) = \frac{(x+3)^{100}}{100} + C.$

7-misol.

$$\int (2x+5)^7 dx = \int (2x+5)^7 \frac{1}{2}\cdot 2dx = \frac{1}{2}\int (2x+7)^7 (2x+7)' dx =$$

$$\frac{1}{2}\int (2x+7)^7 d(2x+7) = \frac{1}{2}\cdot\frac{(2x+7)^8}{8} + C = \frac{(2x+7)^8}{16} + C.$$

8-misol.

$$\int \frac{dx}{3x-2} = \frac{1}{3}\int \frac{3dx}{3x-2} = \frac{1}{3}\int \frac{(3x-2)' dx}{3x-2} = \frac{1}{3}\int \frac{d(3x-2)}{3x-2} = \frac{1}{3}\ln|3x-2| + C.$$

9-misol. $\int \cos 8x dx = \frac{1}{8}\int \cos 8x d(8x) = \frac{1}{8}\sin 8x + C.$

10-misol.

$$\int \sqrt[3]{3x-1}\, dx = \frac{1}{3}\int \sqrt[3]{3x-1}\cdot 3dx = \frac{1}{3}\int (3x-1)^{\frac{1}{3}}\cdot (3x-1)' dx =$$

$$\frac{1}{3}\int (3x-1)^{\frac{1}{3}} d(3x-1) = \frac{1}{3}\cdot\frac{(3x-1)^{\frac{1}{3}+1}}{\frac{1}{3}+1} + C =$$

$$\frac{1}{4}\cdot(3x-1)^{\frac{4}{3}} + C = \frac{1}{4}(3x-1)\sqrt[3]{3x-1} + C.$$

11-misol.

$$\int \frac{dx}{4+9x^2} = \frac{1}{3}\cdot\int \frac{3dx}{4+(3x)^2} = \frac{1}{3}\int \frac{(3x)' dx}{2^2+(3x)^2} = \frac{1}{3}\int \frac{d(3x)}{2^2+(3x)^2} =$$

$$= \frac{1}{3}\cdot\frac{1}{2}\operatorname{arctg}\frac{3x}{2} + C = \frac{1}{6}\operatorname{arctg}\frac{3x}{2} + C.$$

12-misol.

$$\int\frac{\sqrt{5+\ln x}dx}{x}=\int\sqrt{5+\ln x}\cdot\frac{1}{x}\cdot dx=\int\sqrt{5+\ln x}\cdot(5+\ln x)'dx=$$

$$\int(5+\ln x)^{\frac{1}{2}}d(5+\ln x)=\frac{(5+\ln x)^{\frac{1}{2}+1}}{\frac{1}{2}+1}+C=\frac{2}{3}(5+\ln x)^{\frac{3}{2}}+C=$$

$$=\frac{2}{3}(5+\ln x)\sqrt{5+\ln x}+C.$$

13-misol

$$\int e^{-x^2}\cdot xdx=-\frac{1}{2}\int e^{-x^2}(-2x)dx=-\frac{1}{2}\int e^{-x^2}\left(-x^2\right)'dx=$$

$$=-\frac{1}{2}\int e^{-x^2}d\left(-x^2\right)=-\frac{1}{2}e^{-x^2}+C$$

14-misol.

$$\int\frac{2x+3}{x^2+3x-8}dx=\int\frac{(x^2+3x-8)'dx}{x^2+3x-8}=\int\frac{d\left(x^2+3x-8\right)}{x^2+3x-8}=\ln\left|x^2+3x-8\right|+C$$

33.2. O'zgaruvchini almashtirish usuli

Jadvalga kirmagan $\int f(x)dx$ integralni hisoblash talab etilganda bu usulga murojaat qilinadi. $x=\varphi(t)$ almashtirish kiritamiz, bunda $\varphi(t)$ uzluksiz, uzluksiz hosilaga hamda $t=\psi(x)$ teskari funksiyaga ega bo'lsin. U holda $dx=\varphi'(t)dt$ bo'lib,

$$\int f(x)dx=\int f(\varphi(t))\cdot\varphi'(t)dt \qquad (33.1)$$

ekanini isbotlaymiz. Bu tenglikning to'g'riligini ko'rsatish uchun uning o'ng tomonidagi ifodalarning x bo'yicha hosilasi uning chap tomonidagi integral ostidagi $f(x)$ funksiyaga tengligini ko'rsatamiz.

Murakkab funksiyaning hosilasini topish qoidasiga binoan

$$\left(\int f(\varphi(t))\varphi'(t)dt\right)'_x = \left(\int f(\varphi(t))\varphi'(t)dt\right)'_t \cdot t'_x$$

tenglikka ega bo‘lamiz. Aniqmas integralning 1-xossasiga ko‘ra

$$\left(\int f(\varphi(t))\varphi'(t)dt\right)'_t = f(\varphi(t))\cdot\varphi'(t)$$

bo‘lishini hamda $x=\varphi(t)$ *va* $t=\psi(x)$ o‘zaro teskari funksiyalarning hosilalari orasida $t'_x = \frac{1}{x'_t} = \frac{1}{\varphi'(t)}$ munosabat mavjudligini hisobga olsak

$$\left(\int f(\varphi(t))\varphi'(t)dt\right)'_x = f(\varphi(t))\cdot\varphi'(t)\cdot\frac{1}{\varphi'(t)} = f(\varphi(t)) = f(x)$$

kelib chiqadi.

Bu (33.1) tenglikni to‘g‘riligini ko‘rsatadi. (33.1) tenglikning o‘ng tomonidagi integral topilgandan keyin chiqqan natijaga t ning $x=\varphi(t)$ tenglamadan topilgan x orqali qiymati $t=\psi(x)$ ni qo‘yish lozim.

Bundan buyon o‘zgaruvchini almashtirish jarayonini integraldan so‘ng vertikal kesmalar orasiga yozamiz.

Izoh. Ba‘zi hollarda o‘zaruvchini $x=\varphi(t)$ ko‘rinishda emas, balki $t=\psi(x)$ kabi olgan ma‘qul.

Masalan,

$$\int\frac{\psi'(x)dx}{\psi(x)}\left|\begin{array}{l}\psi(x)=t\\ \psi'(x)dx=dt\end{array}\right| = \int\frac{dt}{t} = \ln|t| + C = \ln|\psi(x)| + C.$$

Endi o‘zgaruvchini almashtirib integrallashga bir necha misollar ko‘ramiz.

15-misol.

$$\int\sqrt{a^2-x^2}dx\left|\begin{array}{l}x=a\sin t\\ dx=a(\sin t)'dt=a\cos tdt\end{array}\right| = \int\sqrt{a^2-(a\sin t)^2}\,a\cos tdt =$$

$$\int\sqrt{a^2-a^2\sin^2 t}\,a\cos tdt = \int\sqrt{a^2(1-\sin^2 t)}\,a\cos tdt = a^2\int\sqrt{\cos^2 t}\cos tdt =$$

$$a^2\int\cos^2 tdt = a^2\int\frac{1+\cos 2t}{2}dt = \frac{a^2}{2}\left[\int dt + \int\cos 2tdt\right] = \frac{a^2}{2}\left(t+\frac{1}{2}\sin 2t\right)+C.$$

$x = a\sin t$ formuladan t ni x orqali ifodalaymiz:

$$\frac{x}{a} = \sin t, t = \arcsin\frac{x}{a},$$

$$\sin 2t = 2\sin t\cdot\cos t = 2\sin t\cdot\sqrt{1-\sin^2 t} = 2\cdot\frac{x}{a}\cdot\sqrt{1-\left(\frac{x}{a}\right)^2} =$$

$$= 2\cdot\frac{x}{a}\cdot\frac{\sqrt{a^2-x^2}}{a} = \frac{2x}{a^2}\sqrt{a^2-x^2}.$$

Demak,

$$\int\sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\left(\arcsin\frac{x}{a}+\frac{1}{2}\cdot\frac{2x}{a^2}\sqrt{a^2-x^2}\right)+C =$$

$$= \frac{a^2}{2}\arcsin\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+C$$

16- misol.

$$\int\cos^3 x\sin x dx\left|\begin{array}{l}\cos x = t, d\cos x = dt,\\ (\cos x)'dx = dt, -\sin x dx = dt, \sin x dx = -dt\end{array}\right| =$$

$$= \int t^3(-dt) = -\int t^3 dt = -\frac{t^4}{4}+C = -\frac{\cos^4 x}{4}+C.$$

17-misol.

$$\int(\ln x)^4\frac{dx}{x}\left|\begin{array}{l}\ln x = t, d(\ln x) = dt,\\ (\ln x)'dx = dt, \frac{1}{x}dx = dt\end{array}\right| = \int t^4 dt = \frac{t^5}{5}+C = \frac{\ln^5 x}{5}+C.$$

18-misol.

$$\int \frac{x^2dx}{1+x^6}\left|\begin{array}{l} x^3=t, d(x^3)=dt, (x^3)'dx=dt, \\ 3x^2dx=dt, x^2dx=\frac{1}{3}dt \end{array}\right|=$$

$$=\int \frac{\frac{1}{3}dt}{1+t^2}=\frac{1}{3}\int \frac{dt}{1+t^2}=\frac{1}{3}\text{arctgt}+c=\frac{1}{3}\text{arctg}x^3+C.$$

19-misol.

$$\int e^{\sin x}\cos xdx\left|\begin{array}{l} \sin x=t, d\sin x=dt, \\ (\sin x)'dx=dt, \cos xdx=dt \end{array}\right|=\int e^t dt=e^t+C=e^{\sin x}+C$$

20-misol.

$$\int \frac{dx}{x\sqrt{x^2-1}}\left|x=\frac{1}{t}, dx=\left(\frac{1}{t}\right)'dt=-\frac{dt}{t^2}\right|=\int \frac{-\frac{dt}{t^2}}{\frac{1}{t}\sqrt{\frac{1}{t^2}-1}}=$$

$$=-\int \frac{\frac{dt}{t^2}}{\frac{\sqrt{1-t^2}}{t^2}}=-\int \frac{dt}{\sqrt{1-t^2}}=\arccos t+C$$

$x=\frac{1}{t}$ tenglikdan $t=\frac{1}{x}$ ni hosil qilamiz, natijada

$$\int \frac{dx}{x\sqrt{x^2-1}}=\arccos\frac{1}{x}+C, |x|>1 .$$

33.3. Bo'laklab integrallash usuli

Faraz qilaylik, $u(x)$, $v(x)$ differensiallanuvchi funksiyalar bo'lsin. U holda $d(u\cdot v)=vdu+udv$ yoki $udv=d(uv)-vdu$ bo'lishi ravshan. Oxirgi tenglikni ikkala qismini integrallab

$$\int udv=\int d(uv)-\int vdu \text{ yoki } \int udv=uv-\int vdu \qquad (33.2)$$

bo'laklab integrallash formulasi deb yuritiluvchi formulaga ega bo'lamiz.

Bo'laklab integrallashning mohiyati shundan iboratki, berilgan integralni hisoblashda integral ostidagi $f(x)dx$ ifodani udv ko'paytma shaklida tasvirlab va (33.2) formulani tadbiq qilinsa, berilgan $\int udv$ integralni $\int vdu$ jadval integrali yoki osongina topiladigan integral bilan almashtiriladi. Integrallarni bo'laklab integrallash usuli bilan hisoblashda muhim o'rinni u va dv ifodalarni qanday tanlanishi egallaydi.

Qanday hollarda u, dv larni qanday tanlashni qaraylik.

I. $\int P_n(x)\sin ax dx, \int P_n(x)\cos ax dx, \int P_n(x)e^{ax}dx,$
$\int P_n(x)\mathrm{tg}\, ax dx, \int P_n(x)\mathrm{ctg}\, ax dx$

turdagi integrallarni bo'laklab integrallash uchun $P_n(x)=u$ deb olib qolgan ifodalarni dv orqali belgilash ma'qul, bunda $P_n(x)$ n- darajali ko'phad, a o'zgarmas son.

II. $\int P_n(x)\ln x dx, \int P_n(x)\arcsin ax dx, \int P_n(x)\arccos ax dx,$
$\int P_n(x)\mathrm{arctg}\, ax dx, \int P_n(x)\mathrm{arcctg}\, ax dx$

turdagi integrallarni bo'laklab integrallash uchun transendent ko'paytuvchini u va $P_n(x)dx=dv$ deb olish maqsadga muvofiq.

III. $\int e^{ax}\sin bx dx, \int e^{ax}\cos bx dx$, ($a,b$-o‘zgarmas sonlar) turdagi integrallar uchun esa $e^{ax}=u$ va qolgan ko‘paytuvchilarni dv deb olish ma‘qul.

Bo‘laklab integrallash jarayoniga tegishli belgilashlarni ham xuddi o‘zgaruvchini almashtirish usulidagidek integraldan so‘ng vertikal kesmalar orasiga yozamiz.

21-misol. $\int \ln x dx$ topilsin.

Yechish. $\ln x = u, dx = dv$ desak $(\ln x)' dx = du, \dfrac{dx}{x} = du, v = x$ bo‘lib, (33.2) formulaga binoan

$$\int \ln x dx = x\cdot \ln x - \int x\cdot\frac{dx}{x} = x\cdot\ln x - \int dx = x\ln x - x + C$$

kelib chiqadi.

22- misol.

$$\int xe^x dx \left| \begin{matrix} x = u, e^x dx = dv \\ du = dx, v = \int e^x dx = e^x \end{matrix} \right| = x\cdot e^x - \int e^x dx = x\cdot e^x - e^x + C.$$

23-misol.

$$\int x\sin x dx \left| \begin{matrix} x = u, \sin x dx = dv \\ du = dx, v = \int \sin x dx = -\cos x \end{matrix} \right| =$$

$$= x\cdot(-\cos x) - \int(-\cos x)dx = -x\cos x + \int \cos x dx =$$

$$= -x\cos x + \sin x + C.$$

24-misol.

$$\int \operatorname{arctg} x dx \left| \begin{matrix} \operatorname{arctg} x = u, dx = dv \\ (\operatorname{arctg} x)' dx = du, \dfrac{dx}{1+x^2} = du, v = x \end{matrix} \right| =$$

$$= \operatorname{arctg} x\cdot x - \int \frac{xdx}{1+x^2} = x\cdot\operatorname{arctg} x - \frac{1}{2}\ln\left(1+x^2\right) + C.$$

25-misol.

$$\int\frac{\arcsin x dx}{\sqrt{1+x}}\left|\begin{array}{l}\arcsin x=u, du=(\arcsin x)'dx=\dfrac{dx}{\sqrt{1-x^2}}\\ \dfrac{dx}{\sqrt{1+x}}=dv, v=\int\dfrac{dx}{\sqrt{1+x}}=\int\dfrac{d(1+x)}{\sqrt{1+x}}=2\sqrt{1+x}\end{array}\right|=$$

$$=\arcsin x\cdot 2\sqrt{1+x}-\int 2\sqrt{1+x}\cdot\frac{dx}{\sqrt{1-x^2}}=$$

$$=2\sqrt{1+x}\cdot\arcsin x-2\int\sqrt{1+x}\cdot\frac{dx}{\sqrt{1+x}\cdot\sqrt{1-x}}=2\sqrt{1+x}\arcsin x+2\int\frac{d(1-x)}{\sqrt{1-x}}=$$

$$=2\sqrt{1+x}\arcsin x+4\sqrt{1-x}+C.$$

26-misol

$$J=\int e^{\alpha x}\cos\beta x dx\left|\begin{array}{l}e^{\alpha x}=u, du=\alpha e^{\alpha x}dx,\\ \cos\beta dx=dv, v=\int\cos\beta x dx=\dfrac{1}{\beta}\sin\beta x\end{array}\right|=$$

$$=e^{\alpha x}\cdot\frac{1}{\beta}\sin\beta x-\int\frac{1}{\beta}\sin\beta x\cdot\alpha e^{\alpha x}dx=\frac{1}{\beta}e^{\alpha x}\sin\beta x-\frac{\alpha}{\beta}\int e^{\alpha x}\sin\beta x dx\left|\begin{array}{l}e^{\alpha x}=u, du=\alpha e^{\alpha x}\\ \sin\beta x dx=dv, v=-\dfrac{\cos\beta x}{\beta}\end{array}\right|=$$

$$=\frac{1}{\beta}e^{\alpha x}\sin\beta x-\frac{\alpha}{\beta}\left(-\frac{1}{\beta}e^{\alpha x}\cos\beta x+\frac{\alpha}{\beta}\int e^{\alpha x}\cos\beta x dx\right)=$$

$$=\frac{1}{\beta}e^{\alpha x}\sin\beta x+\frac{\alpha}{\beta^2}e^{\alpha x}\cos\beta x-\frac{\alpha^2}{\beta^2}\int e^{\alpha x}\cos\beta x dx.$$

Shunday qilib,

$$J=\frac{1}{\beta}e^{\alpha x}\sin\beta x+\frac{\alpha}{\beta^2}e^{\alpha x}\cos\beta x-\frac{\alpha^2}{\beta^2}\cdot J$$

tenglikka ega bo'ldik. Bu tenglamani berilgan J integralga nisbatan yechib uni topamiz.

$$J+\frac{\alpha^2}{\beta^2}\cdot J=\frac{\beta e^{\alpha x}\sin\beta x+\alpha e^{\alpha x}\cos\beta x}{\beta^2},$$

$$\frac{\beta^2+\alpha^2}{\beta^2}\cdot J=\frac{e^{\alpha x}(\beta\sin\beta x+\alpha\cos\beta x)}{\beta^2},$$

$$J = \frac{e^{\alpha x}}{\alpha^2+\beta^2}(\beta \sin \beta x + \alpha \cos \beta x).$$

Demak,

$$\int e^{\alpha x} \cos \beta x dx = \frac{e^{\alpha x}}{\alpha^2+\beta^2}(\beta \sin \beta x + \alpha \cos \beta x) + C.$$

O‘z-o‘zini tekshirish uchun savollar

1. Bevosita integrallash usuli nimaga asoslangan.
2. O‘zgaruvchini almashtirish formulasini isbotlang.
3. Bo‘laklab integrallash formulasini isbotlang.
4. Bo‘laklab integrallashda muhim o‘rinni nima egallaydi.
5. Ko‘phad bilan trigonometrik funksiyalar yoki ko‘rsatgichli funksiyaning ko‘paytmasi ishtirok etgan integrallarni bo‘laklab integrallashda *u, dv* lar qanday tanlanadi?
6. Ko‘phad bilan teskari trigonometrik funksiyalar yoki logarifmik funksiyaning ko‘paytmasi ishtirok etgan integrallarni bo‘laklab integrallashda *u, dv* lar qanday tanlanadi?
7. Ko‘rsatgichli va trigonometrik funksiyalarning ko‘paytmasi ishtirok etgan integrallarni bo‘laklab integrallashda *u, dv* lar qanday tanlanadi?
8. Bo‘laklab integrallash jarayoniga tegishli belgilashlar qaerga yoziladi?

Mustaqil yechish uchun mashqlar

Quyidagi integrallar topilsin.

1. $\int (x^3 + 3\sin x - 8)dx$. javob: $\frac{x^4}{4} - 3\cos x - 8x + C$.
2. $\int \frac{x^3+4x+2}{2x}dx$. javob: $\frac{x^3}{6} + 2x + \ln|x| + C$.*(x≠0)*
3. $\int x^3\sqrt{3-2x^4}dx$. javob: $-\frac{1}{12}\cdot\sqrt{(3-2x^4)^3} + C$.
4. $\int \frac{(\arcsin x)^2 dx}{\sqrt{1-x^2}}$. javob: $\frac{1}{3}(\arcsin x)^3 + C$.
5. $\int \frac{(2x-5)dx}{x^2-5x+6}$. javob: $\ln|x^2-5x+6| + C$. *(x≠2, x≠3).*

6. $\int \frac{\sin x dx}{\sqrt[4]{\cos^3 x}}$. javob: $-4\sqrt[4]{\cos x} + C$.

7. $\int \sin^4 x \sin 2x dx$. javob: $\frac{1}{3}\sin^6 x + C$.

8. $\int \frac{dx}{\sin^2 x\sqrt{1 + ctgx}}$. javob: $-2\sqrt{1 + ctgx} + C$.

9. $\int e^{-\cos x} \sin x dx$. javob: $e^{-\cos x} + C$.

10. $\int \frac{x^2 dx}{\sqrt{x^6 + 3}}$. javob: $\frac{1}{3}\ln\left|x^3 + \sqrt{x^6 + 3}\right| + C$.

11. $\int \frac{e^x dx}{e^{2x} + 4}$. javob: $\frac{1}{2} arctg\frac{e^x}{2} + C$.

12. $\int \frac{dx}{x\sqrt[3]{\ln x}}$. javob: $\frac{3}{2}\sqrt[3]{\ln^2 x} + C$.

13. $\int \frac{dx}{x^2\sqrt{1 + x^2}}$. javob: $C - \frac{\sqrt{1+x^2}}{x}$.

Ko'rsatma. $x=tgt$ almashtirish olinsin.

14. $\int \arccos x dx$. javob: $x\arccos x - \sqrt{1 - x^2} + C$.

15. $\int \ln(x^2 + 1)dx$. javob: $x\ln(x^2 + 1) - 2(x - arctgx) + C$.

16. $\int \frac{\text{xarctgx}}{\sqrt{1 + x^2}}dx$. javob: $\sqrt{1+x^2}\text{arctgx} - \ln\left|x + \sqrt{1+x^2}\right| + C$.

17. $\int x^2 e^{-x} dx$. javob: $-e^{-x}(x^2 + 2x + 2) + C$.

18. $\int e^{2x}\cos 3x dx$. javob: $\frac{e^{2x}}{13}(3\sin 3x + 2\cos 3x) + C$.

19. $\int \text{xarcctgxdx}$. javob: $\frac{1}{2}(x^2\text{arcctgx} + x - \text{arcctgx}) + C$.

20. $\int \frac{x^3 dx}{\sqrt{9 - x^2}}$. javob: $-\frac{(x^2 + 18)\sqrt{9 - x^2}}{3} + C$.

34. KVADRAT UCHHAD QATNASHGAN BA'ZI-BIR FUNKSIYALARNI VA ENG SODDA RATSIONAL KASRLARNI HAMDA RATSIONAL FUNKSIYANI INTEGRALLASH

34.1. Kvadrat uchhad qatnashgan ba'zi-bir funksiyalarni integrallash

1. $J_1 = \int \frac{dx}{ax^2 + bx + c}$ integralni hisoblash talab etilsin, bunda $a \neq 0, b, c$ o'zgarmas sonlar. Bu integralni integrallashga kirishishdan oldin $ax^2 + bx + c$ kvadrat uchhadni kvadratlar yig'indisi yoki kvadratlar ayirmasi shakliga keltiramiz, ya'ni kvadrat uchhaddan to'liq kvadratni ajratamiz:

$$ax^2 + bx + c = a(x^2 + \frac{b}{a}x + \frac{c}{a}) = a(x^2 + 2\cdot\frac{b}{2a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a}) =$$

$$= a\left((x + \frac{b}{2a})^2 - \frac{b^2 - 4ac}{4a^2}\right).$$

$\frac{4ac - b^2}{4a^2} = \pm k^2$ belgilashni kiritsak.

$$ax^2 + bx + c = a\left[(x + \frac{b}{2a})^2 \pm k^2\right]$$

tenglikka ega bo'lamiz.

$b^2 - 4ac = 0$, ya'ni kvadrat uchhad ikki karrali ildizga ega bo'lsin. U holda $k=0$ bo'lib berilgan integral osongina hisoblanadi:

$$J_1 = \int \frac{dx}{ax^2 + bx + c} = \frac{1}{a}\int \frac{dx}{(x + \frac{b}{2a})^2} \left| \begin{matrix} x + \frac{b}{2a} = t \\ dx = dt \end{matrix} \right| =$$

$$= \frac{1}{a}\int \frac{dt}{t^2} = -\frac{1}{a}\cdot\frac{1}{t} + C = -\frac{1}{a(x + \frac{b}{2a})} + C.$$

Integrallash jarayonida integrallar jadvalidagi 4-formuladan foydalanildi.

$b^2-4ac\neq 0$ bo'lsin. U holda berilgan integral

$$J_1=\int\frac{dx}{ax^2+bx+c}=\frac{1}{a}\int\frac{dx}{(x+\frac{b}{2a})^2\pm k^2}\left|\begin{matrix}x+\frac{b}{2a}=t\\ dx=dt\end{matrix}\right|=\frac{1}{a}\int\frac{dt}{t^2\pm k^2}$$

ko'rinishni oladi. Bu yerdagi k^2 oldidagi plyus ishora ax^2+bx+c kvadrat uchhad kompleks ildizlarga ega bo'lganda ($b^2-4ac<0$), minus ishora esa kvadrat uchhad haqiqiy har xil ($b^2-4ac>0$) ildizlarga ega bo'lganda olinadi.

Oxirgi integral jadval integraldir. U integrallar jadvalidagi 16- yoki 20- formulalarning birortasi yordamida topiladi.

1-misol. Ushbu $\int\frac{dx}{x^2-4x+4}$ integral hisoblansin.

Yechish. $\int\frac{dx}{x^2-4x+4}=\int\frac{dx}{(x-2)^2}=\int\frac{d(x-2)}{(x-2)^2}=-\frac{1}{x-2}+C.$

Bu yerda integrallar jadvalidagi 4-formuladan foydalandik.

2-misol. Ushbu $\int\frac{dx}{x^2-6x+5}$ integral hisoblansin.

Yechish.

$$\int\frac{dx}{x^2-6x+5}=\int\frac{dx}{x^2-6x+9-4}=\int\frac{dx}{(x-3)^2-2^2}\left|\begin{matrix}x-3=t\\ dx=dt\end{matrix}\right|=\int\frac{dt}{t^2-2^2}=$$

$$=\frac{1}{2\cdot 2}\ln\left|\frac{t-2}{t+2}\right|+C=\frac{1}{4}\ln\left|\frac{x-3-2}{x-3+2}\right|+C=\frac{1}{4}\ln\left|\frac{x-5}{x-1}\right|+C.$$

3-misol. Ushbu $\int\frac{dx}{3x^2-2x+4}$ integral topilsin.

Yechish.

$$\int\frac{dx}{3(x^2-\frac{2}{3}x+\frac{4}{3})}=\frac{1}{3}\int\frac{dx}{x^2-2\cdot\frac{1}{3}x+\frac{1}{9}-\frac{1}{9}+\frac{4}{3}}=\frac{1}{3}\int\frac{dx}{(x-\frac{1}{3})^2+\frac{11}{9}}=$$

$$\frac{1}{3}\int\frac{d(x-\frac{1}{3})}{(x-\frac{1}{3})^2+(\frac{\sqrt{11}}{3})^2}=\frac{1}{3}\cdot\frac{1}{\frac{\sqrt{11}}{3}}\operatorname{arctg}\frac{x-\frac{1}{3}}{\frac{\sqrt{11}}{3}}+C=\frac{1}{\sqrt{11}}\operatorname{arctg}\frac{3x-1}{\sqrt{11}}+C.$$

Bu yerda integrallar jadvalidagi 16-formuladan foydalanildi.

II. Ushbu

$$J_2=\int\frac{Ax+B}{ax^2+bx+c}dx$$

integralni qaraymiz, bunda A, B, $a\neq 0$, b, c ma'lum o'zgarmas sonlar va ax^2+bx+c kvadrat uchhad ikki karrali ildizga ega emas.

Yechish. Integrallashni amalga oshirish uchun integral ostidagi kasrni boshqacha ko'rinishda ya'ni suratda maxrajning hosilasi $(ax^2+bx+c)'=2ax+b$ ni ajratib yozamiz.

$$J_2=\int\frac{Ax+B}{ax^2+bx+c}dx=\int\frac{\frac{A}{2a}(2ax+b)+(B-\frac{Ab}{2a})}{ax^2+bx+c}dx=$$

$$\frac{A}{2a}\int\frac{2ax+b}{ax^2+bx+c}dx+(B-\frac{Ab}{2a})\int\frac{dx}{ax^2+bx+c}=$$

$$\frac{A}{2a}\int\frac{(ax^2+bx+c)'dx}{ax^2+bx+c}+(B-\frac{Ab}{2a})\cdot J_1=\frac{A}{2a}\int\frac{d(ax^2+bx+c)}{(ax^2+bx+c)}+(B-\frac{Ab}{2a})\cdot J_1=$$

$$\frac{A}{2a}\ln\left|ax^2+bx+c\right|+(B-\frac{Ab}{2a})\cdot J_1.$$

Bu yerda $J_1=\int\frac{dx}{ax^2+bx+c}$ integralni hisoblashni bilamiz.

Shunday qilib, integrallar jadvalidagi 6-formulaga binoan

$$J_2=\frac{A}{2a}\ln\left|ax^2+bx+c\right|+(B-\frac{Ab}{2a})\cdot J_1$$

tenglikka ega bo'ldik.

4-misol. Ushbu $\int \frac{(3x-2)}{5x^2-3x+2}dx$ integral topilsin.

Yechish.

$$\int \frac{(3x-2)}{5x^2-3x+2}dx = \int \frac{\frac{3}{10}(10x-3)-2+\frac{9}{10}}{5x^2-3x+2}dx = \frac{3}{10}\int \frac{10x-3}{5x^2-3x+2}dx - \frac{11}{10}\int \frac{dx}{5x^2-3x+2} =$$

$$= \frac{3}{10}\int \frac{(5x^2-3x+2)'dx}{5x^2-3x+2} - \frac{11}{10}\int \frac{dx}{5(x^2-\frac{3}{5}x+\frac{2}{5})} = \frac{3}{10}\ln\left|5x^2-3x+2\right| - \frac{11}{50}\int \frac{dx}{(x^2-2\cdot\frac{3}{10}x+\frac{9}{100})+\frac{2}{5}-\frac{9}{100}} =$$

$$= \frac{3}{10}\ln\left|5x^2-3x+2\right| - \frac{11}{50}\int \frac{dx}{(x-\frac{3}{10})^2+\frac{31}{100}} = \frac{3}{10}\ln\left|5x^2-3x+2\right| - \frac{11}{50}\int \frac{d(x-\frac{3}{10})}{(x-\frac{3}{10})^2+(\frac{\sqrt{31}}{10})^2} =$$

$$= \frac{3}{10}\ln\left|5x^2-3x+2\right| - \frac{11}{50}\cdot\frac{1}{\frac{\sqrt{31}}{10}}\operatorname{arctg}\frac{x-\frac{3}{10}}{\frac{\sqrt{31}}{10}} + C = \frac{3}{10}\ln\left|5x^2-3x-2\right| - \frac{11}{5\sqrt{31}}\operatorname{arctg}\frac{10x-3}{\sqrt{31}} + C$$

Integrallash jarayonida $\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C$ formuladan hamda integrallar jadvalidagi 16-formuladan foydalandik.

III. Ushbu $\int \frac{dx}{\sqrt{ax^2+bx+c}}$ integralni qaraymiz, bunda $a\neq 0, b, c$ ma'lum o'zgarmas sonlar va ax^2+bx+c kvadrat uchhad ikki karrali ildizga ega emas.

Bu integral I-bandda bajarilgan amallar yordamida $a>0$ bo'lganda $\int \frac{dt}{\sqrt{t^2 \pm k^2}}$ va $a<0$ bo'lganda $\int \frac{dt}{\sqrt{k^2-t^2}}$ ko'rinishdagi jadval integrallaridan biriga keladi.

5-misol. Ushbu $\int \frac{dx}{\sqrt{2-3x-x^2}}$ integral topilsin.

Yechish. Avval ildiz ostidagi kvadrat uchhadni kvadratlar yig'indisi yoki kvadratlar ayirmasi ko'rinishiga keltiramiz:

$$2-3x-x^2=2-(3x+x^2)=2-(x^2+3x)=2-(x^2+2\cdot\frac{3}{2}x+(\frac{3}{2})^2-\frac{9}{4})=$$

$$=2-(x^2+2\cdot\frac{3}{2}x+(\frac{3}{2})^2)+\frac{9}{4}=2+\frac{9}{4}-(x+\frac{3}{2})^2=\frac{17}{4}-(x+\frac{3}{2})^2=(\frac{\sqrt{17}}{2})^2-(x+\frac{3}{2})^2.$$

Shunday qilib, berilgan integral

$$\int\frac{dx}{\sqrt{2-3x-x^2}}=\int\frac{dx}{\sqrt{(\frac{\sqrt{17}}{2})^2-(x+\frac{3}{2})^2}}=\int\frac{d(x+\frac{3}{2})}{\sqrt{(\frac{17}{2})^2-(x+\frac{3}{2})^2}}$$

ko'rinishni oladi. Bundan integrallar jadvaliidagi 18-formuladan foydalanib quyidagini hosil qilamiz:

$$\int\frac{dx}{\sqrt{2-3x-x^2}}=\arcsin\frac{x+\frac{3}{2}}{\frac{\sqrt{17}}{2}}+C=\arcsin\frac{2x+3}{\sqrt{17}}+C.$$

IV. Ushbu $\int\frac{Ax+B}{\sqrt{ax^2+bx+c}}dx$ integralani qaraymiz, bunda A,B,a,b,c ma'lum o'zgarmas sonlar va ax^2+bx+c kvadrat ushhad ikki karrali ildizga ega emas. Bu integral II-bandda bajarilgan almashtirishlar yordamida hisoblanadi:

$$\int\frac{Ax+B}{\sqrt{ax^2+bx+c}}dx=\int\frac{\frac{A}{2a}(2ax+b)+(B-\frac{Ab}{2a})}{\sqrt{ax^2+bx+c}}dx=$$

$$=\frac{A}{2a}\int\frac{2ax+b}{\sqrt{ax^2+bx+c}}dx+(B-\frac{Ab}{2a})\int\frac{dx}{\sqrt{ax^2+bx+c}}.$$

Hosil bo'lgan integraldan ikkinchisi III-bandda qaralgan integral bo'lgani uchun uni hisoblashni bilamiz.

Shuning uchun birinchi integralni hisoblashga kirishamiz:

$$\int\frac{2ax+b}{\sqrt{ax^2+bx+c}}dx=\int\frac{(ax^2+bx+c)'dx}{\sqrt{ax^2+bx+c}}=$$

$$=\int\frac{d(ax^2+bx+c)}{\sqrt{ax^2+bx+c}}=2\sqrt{ax^2+bx+c}+C.$$

Bunda integrallar jadvalidagi 5-formuladan foydalanildi.

6-misol. Ushbu $\int\frac{(x+3)dx}{\sqrt{3+4x-4x^2}}$ integral topilsin.

Yechish.

$$\int\frac{(x+3)dx}{\sqrt{3+4x-4x^2}}=\int\frac{-\frac{1}{8}(4-8x)+\frac{7}{2}}{\sqrt{3+4x-4x^2}}dx=-\frac{1}{8}\int\frac{(4-8x)}{\sqrt{3+4x-4x^2}}dx+\frac{7}{2}\int\frac{dx}{\sqrt{3-(4x^2-4x)}}=$$

$$=-\frac{1}{8}\int\frac{(3+4x-4x^2)'dx}{\sqrt{3+4x-4x^2}}+\frac{7}{2}\int\frac{dx}{\sqrt{4-(4x^2-4x+1)}}=-\frac{1}{8}\int\frac{d(3+4x^2-4x^2)}{\sqrt{3+4x-4x^2}}+\frac{7}{2}\int\frac{dx}{\sqrt{2^2-(2x-1)^2}}=$$

$$=-\frac{1}{8}\cdot 2\sqrt{3+4x-4x^2}+\frac{7}{4}\int\frac{d(2x-1)}{\sqrt{2^2-(2x-1)^2}}=-\frac{1}{4}\sqrt{3+4x-4x^2}+\frac{7}{4}\arcsin\frac{2x-1}{2}+C.$$

Shuni aytish joizki qaralgan integrallarda ax^2+bx+c kvadrat uchhad ikki karralli ildizga ega bo'lganda ham integrallar I-bandda bajarilgan almashtirishlar yordamida yanada osonroq hisoblanadi.

34.2. Eng sodda ratsional kasrlarni integrallash

Ta'rif. I. $\frac{A}{x-a}$, II. $\frac{A}{(x-a)^k}$, III. $\frac{Ax+B}{x^2+px+q}$,

IV. $\frac{Ax+B}{(x^2+px+q)^k}$ ko'rinishdagi kasrlar mos ravishda I-, II-, III- va IV- tur eng sodda ratsional kasrlar deyiladi, bunda $,k\geq 2$ natural son, A,B,a,p,q o'zgarmas haqiqiy sonlar bo'lib x^2+px+q kvadrat uchhad haqiqiy ildizlarga ega emas, ya'ni $q-\frac{p^2}{4}>0$.

Shu kasrlarning integrallarini topamiz. I- va II- tur eng sodda kasrlarni integrallash jadval integrallari yordamida amalga oshiriladi, ya'ni

$$\int \frac{Adx}{x-a} = A\int \frac{(x-a)'dx}{x-a} = A\int \frac{d(x-a)}{x-a} = A\ln|x-a| + C,$$

$$\int \frac{Adx}{(x-a)^k} = A\int (x-a)^{-k} dx = A\int (x-a)^{-k} d(x-a) =$$

$$A\frac{(x-a)^{-k+1}}{-k+1} + C = -\frac{A}{(k-1)(x-a)^{k-1}} + C$$

Endi III-tur eng sodda ratsional kasrni integrallaymiz. Bu integrani hisoblash usuli bilan kvadrat uchhad qatnashgan funksiyani integrallash jarayonida tanishgan edik.

$$\int \frac{Ax+B}{x^2+px+q}dx = \int \frac{Ax+B}{(x+\frac{p}{2})^2 + q - \frac{p^2}{4}}dx \left| \begin{matrix} x+\frac{p}{2} = t, x = t-\frac{p}{2} \\ dx = dt, q-\frac{p^2}{4} = a^2 \end{matrix} \right| =$$

$$\int \frac{A(t-\frac{p}{2})+B}{t^2+a^2}dt = \int \frac{At+(B-\frac{Ap}{2})}{t^2+a^2}dt = A\int \frac{tdt}{t^2+a^2} + \left(B-\frac{Ap}{2}\right)\cdot$$

$$\int \frac{dt}{t^2+a^2} = \frac{A}{2}\ln(t^2+a^2) + \left(B-\frac{Ap}{2}\right)\frac{1}{a}\text{arctg}\frac{t}{a} + C =$$

$$\frac{A}{2}\ln(x^2+px+q) - \left(B-\frac{Ap}{2}\right)\frac{1}{\sqrt{q-\frac{p^2}{4}}}\text{arctg}\frac{x+\frac{p}{2}}{\sqrt{q-\frac{p^2}{4}}} + C = \frac{A}{2}\ln(x^2+px+q) +$$

$$+\left(B-\frac{Ap}{2}\right)\cdot\frac{2}{\sqrt{4q-p^2}}\text{arctg}\frac{2x+p}{\sqrt{4q-p^2}} + C.$$

Endi IV- turdagi eng sodda ratsional kasrni integrallashga kirishamiz.

$$\int\frac{Ax+B}{(x^2+px+q)^k}dx\left|\begin{matrix}x+\frac{p}{2}=t, x=t-\frac{p}{2}\\ dx=dt, q-\frac{p^2}{4}=a^2\end{matrix}\right|=A\int\frac{tdt}{\left(t^2+a^2\right)^k}+\left(B-\frac{Ap}{2}\right)\int\frac{dt}{\left(t^2+a^2\right)^k}=$$

$$=\frac{A}{2}\int\left(t^2+a^2\right)^{-k}d\left(t^2+a^2\right)+\left(B-\frac{Ap}{2}\right)\int\frac{dt}{\left(t^2+a^2\right)^k}=$$

$$=\frac{A}{2}\cdot\frac{\left(t^2+a^2\right)^{-k+1}}{-k+1}+\left(B-\frac{Ap}{2}\right)\int\frac{dt}{\left(t^2+a^2\right)^k}. \qquad (34.1)$$

$J_k=\int\frac{dt}{\left(t^2+a^2\right)^k}$ integralni topamiz.

$$J_k=\int\frac{dt}{\left(t^2+a^2\right)^k}\left|\frac{1}{\left(t^2+a^2\right)^k}=u, dt=dv, du=\left(\left(t^2+a^2\right)^{-k}\right)'dt=-\frac{2ktdt}{\left(t^2+a^2\right)^{k+1}}, v=t, k\in N\right|=$$

$$=\frac{t}{\left(t^2+a^2\right)^k}+2k\int\frac{t^2dt}{\left(t^2+a^2\right)^{k+1}}=\frac{t}{\left(t^2+a^2\right)^k}+2k\int\frac{t^2+a^2-a^2}{\left(t^2+a^2\right)^{k+1}}dt=$$

$$=\frac{t}{\left(t^2+a^2\right)^k}+2k\int\frac{dt}{\left(t^2+a^2\right)^k}-2ka^2\int\frac{dt}{\left(t^2+a^2\right)^{k+1}}.$$

Bu yerda bo‘laklab integrallash formulasidan foydalanildi. Natijada

$$J_k=\frac{t}{\left(t^2+a^2\right)^k}+2k\cdot J_k-2ka^2J_{k+1}$$

tenglikka ega bo‘ldik. Bu tenglikdan J_{K+1} ni topamiz:

$$2ka^2J_{k+1}=\frac{t}{\left(t^2+a^2\right)^k}+2k\cdot J_k-J_k \text{ yoki } J_{k+1}=\frac{t}{2ka^2\left(t^2+a^2\right)^k}+\frac{2k-1}{2ka^2}J_k. \quad (34.2)$$

(34.2) munosabat **rekkurent** formula deyiladi. $k\geq 1$ bo‘lganda J_{k+1} integral shu formuladan foydalanib topiladi. Masalan,

$$J_2=\int\frac{dt}{(t^2+a^2)^2}=\frac{t}{2\cdot1\cdot a^2(t^2+a^2)}+\frac{2\cdot1-1}{2\cdot1\cdot a^2}\cdot J_1=\frac{t}{2a^2(t^2+a^2)}+\frac{1}{2a^2}\int\frac{dt}{t^2+a^2}=$$

$$=\frac{t}{2a^2(t^2+a^2)}+\frac{1}{2a^2}\cdot\frac{1}{a}\text{arctg}\frac{t}{a}+C.$$

Rekkurent formuladan foydalanib J_k ni J_{k-1} orqali ifodalaymiz, so'ngra J_{k-1} ni J_{k-2} orqali ifodalaymiz va hokazo bu jarayonni

$$J_1=\int\frac{dt}{t^2+a^2}=\frac{1}{a}\text{arctg}\frac{t}{a}+C$$

integral hosil bo'lguncha davom ettirib J_k integralni aniqlaymiz. J_k integralning topilgan qiymatini (34.1) ga qo'yib, chiqqan natijaga t o'rniga $x+\frac{p}{2}$ ni va a o'rniga $\sqrt{q-\frac{p^2}{4}}$ ni qo'ysak IV-tur eng sodda ratsional kasrning qiymati hosil bo'ladi.

Shunday qilib barcha turdagi eng sodda ratsional kasrlarning integrallari mavjud ekan.

7-misol. $\int\frac{3x+5}{x^2+2x+10}dx$ integral topilsin.

Yechish.

$$\int\frac{3x+5}{x^2+2x+10}dx=\int\frac{3x+5}{(x+1)^2+3^2}\left|\begin{array}{l}x+1=t, x=t-1\\ dx=dt\end{array}\right|=\int\frac{3(t-1)+5}{t^2+3^2}dt=$$

$$=\int\frac{3t+2}{t^2+3^2}dt=3\int\frac{tdt}{t^2+3^2}+2\int\frac{dt}{t^2+3^2}=$$

$$=\frac{3}{2}\ln(t^2+3^2)+\frac{2}{3}\text{arctg}\frac{t}{3}+C=\frac{3}{2}\ln(x^2+2x+10)+\frac{2}{3}\text{arctg}\frac{x+1}{3}+C.$$

8- misol. $\int\frac{x-1}{(x^2+2x+2)^2}dx$ topilsin.

Yechish.

$$\int\frac{x-1}{(x^2+2x+2)^2}dx=\int\frac{(x-1)dx}{\left[(x+1)^2+1\right]^2}\left|\begin{matrix}x+1=t, x=t-1,\\ dx=dt\end{matrix}\right|=\int\frac{(t-1-1)dt}{\left(t^2+1\right)^2}=\int\frac{tdt}{\left(t^2+1\right)^2}-2\int\frac{dt}{\left(t^2+1\right)^2}$$

$$\int\frac{tdt}{\left(t^2+1\right)^2}=\frac{1}{2}\int\frac{\left(t^2+1\right)'dt}{\left(t^2+1\right)^2}=\frac{1}{2}\int\frac{d\left(t^2+1\right)}{\left(t^2+1\right)^2}=-\frac{1}{2}\cdot\frac{1}{t^2+1}=-\frac{1}{2\left(x^2+2x+2\right)^2}.$$

(34.2) rekkurent formulaga asosan *(a=1, κ=1)*

$$\int\frac{dt}{\left(t^2+1\right)^2}=\frac{t}{2\cdot1\cdot1^2\left(t^2+1\right)}+\frac{2\cdot1-1}{2\cdot1^2}\int\frac{dt}{t^2+1}=\frac{t}{2\left(t^2+1\right)}+$$

$$+\frac{1}{2}\text{arctg}t=\frac{x+1}{2\left(x^2+2x+2\right)}+\frac{1}{2}\cdot\text{arctg}(x+1)$$

kelib chiqadi. Integrallarning topilgan qiymatlarini ularni o‘rniga qo‘yib berilgan integralni topamiz:

$$\int\frac{(x-1)dx}{\left(x^2+2x+2\right)^2}=-\frac{1}{2\left(x^2+2x+2\right)}-\frac{x+1}{x^2+2x+2}-\text{arctg}(x+1)+C=$$

$$=-\frac{2x+3}{x^2+2x+2}-\text{arctg}(x+1)+C.$$

9-misol. $J_3=\int\frac{dx}{\left(1+x^2\right)^3}$ topilsin.

Yechish.(34.2) rekkurent formulaga binoan:

$$J_3=\frac{x}{2\cdot2\cdot1^2\left(x^2+1\right)^2}+\frac{2\cdot2-1}{2\cdot2\cdot1^2}J_2=\frac{x}{4\left(x^2+1\right)^2}+\frac{3}{4}\left(\frac{x}{2\cdot1\cdot1^2\left(x^2+1\right)}+\frac{2\cdot1-1}{2\cdot1^2}\cdot J_1\right)=$$

$$=\frac{x}{4\left(x^2+1\right)^2}+\frac{3}{4}\left(\frac{x}{2\cdot\left(x^2+1\right)}+\frac{1}{2}\int\frac{dx}{x^2+1}\right)=\frac{x}{4(x^2+1)^2}+\frac{3}{4}\left(\frac{x}{2\left(1+x^2\right)}+\frac{1}{2}\text{arctg}x\right)+C.$$

Shunday qilib,

$$\int\frac{dx}{\left(1+x^2\right)^3}=\frac{1}{4}\left(\frac{x}{\left(1+x^2\right)^2}+\frac{3}{2}\cdot\frac{x}{\left(1+x^2\right)}+\frac{3}{2}\text{arctg}x\right)+C.$$

34.3. Kasr-ratsional funksiyani eng sodda ratsional kasrlarga yoyish

Ma'lumki,

$$P_n(x) = a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n$$

funksiya n-darajali ko'phad deyiladi, bunda $a_0 \neq O, a_1, a_2, \ldots a_n$ o'zgarmas haqiqiy sonlar ko'phadning koeffitsientlari, n-natural son esa daraja ko'rsatkichi.

Ta'rif. Ikki ko'phadning nisbati kasr-ratsional funksiya yoki ratsional kasr deyiladi:

$$f(x) = \frac{Q_m(x)}{P_n(x)} = \frac{b_0x^m + b_1x^{m-1} + \ldots + b_{m-1}x + b_m}{a_0x^n + a_1x^{n-1} + \ldots + a_{n-1}x + a_n}.$$

Agar $m < n$ bo'lsa, u holda ratsional kasr to'g'ri, $m \geq n$ bo'lganda ratsional kasr noto'g'ri kasr deyiladi.

Masalan $\frac{3x+1}{x^3+8}$ - to'g'ri kasr, $\frac{x^2+1}{2x^2+x+1}$ va $\frac{x^3+3x+1}{x^2+x+1}$ kasrlar noto'g'ri kasrlardir. $f(x)$ ratsional kasr noto'g'ri kasr bo'lganda kasrning $Q_m(x)$ suratini uning $P_n(x)$ maxrajiga bo'lib kasrni

$$\frac{Q_m(x)}{P_n(x)} = q_k(x) + \frac{r(x)}{P_n(x)}$$

ko'rinishga keltiriladi, bunda $q_k(x)$-k=m-n-darajali ko'phad, $\frac{r(x)}{P_n(x)}$ to'g'ri kasr.

Shunday qilib, noto'g'ri ratsional kasrni ko'phad (u nolinchi darajali bo'lishi ham mumkin) bilan to'g'ri kasrning yig'indisi ko'rinishida tasvirlash mumkin ekan.

$q_k(x)=c_0x^k+c_1x^{k-1}+\ldots+c_k$ ko'phadning integrali

$$\int q_k(x)dx = \int\left(c_0x^k + c_1x^{k-1} + \ldots + c_k\right)dx = c_0\frac{x^{k+1}}{k+1} + c_1\frac{x^k}{k} + \ldots + c_kx + C$$

kabi topilgani uchun noto'g'ri kasrni integrallash to'g'ri kasrni integrallashga keltiriladi.

Ushbu

$$f(x)=\frac{Q_m(x)}{P_n(x)}$$

to‘g‘ri ratsional kasrni qaraymiz. Kasrning maxraji

$$P_n(x)=a_0(x-\alpha_1)^{k_1}\dots(x-\alpha_r)^{k_r}\left(x^2+p_1x+q_1\right)^{s_1}\cdot$$
$$\cdot\left(x^2+p_2x+q_2\right)^{s_2}\dots\left(x^2+p_ex+q_e\right)^{s_e} \quad (34.3)$$

ko‘rinishdagi ko‘paytuvchilarga ajralsin.

Quyidagi teorema o‘rinli:

34.1-teorema. Maxraji (34.3) ko‘rinishdagi yoyilmaga ega bo‘lgan har qanday

$$f(x)=\frac{Q_m(x)}{P_n(x)}$$

to‘g‘ri ratsional kasrni I-, II-, III-, IV- turdagi eng sodda ratsional kasrlarning yig‘indisi ko‘rinishda tasvirlash mumkin. Bunda:

a) (34.3) yoyilmaning $x-\alpha$ ko‘rinishdagi ko‘paytuvchisiga I-turdagi bitta eng sodda

$$\frac{A_1}{x-\alpha}$$

kasr mos keladi;

b) (34.3) yoyilmaning $(x-\alpha)^k$ ko‘rinishdagi ko‘paytuvchisiga I- va II- turdagi κ ta eng sodda kasrlarning yig‘indisi

$$\frac{A_1}{x-\alpha}+\frac{A_2}{(x-\alpha)^2}+\frac{A_3}{(x-\alpha)^3}+\dots+\frac{A_\kappa}{(x-\alpha)^\kappa}$$

mos keladi;

c) (34.3)yoyilmaning x^2+px+q ko‘rinishidagi ko‘paytuvchisiga

$$\frac{A_1x+B_1}{x^2+px+q}$$

III-turdagi bitta eng sodda ratsional kasr mos keladi;

d) (34.3)yoyilmaning $(x^2+px+q)^s$ ko'rinishdagi ko'paytuvchisiga III- va- IV turdagi s ta eng sodda ratsional kasrlarning yig'indisi

$$\frac{A_1x+B_1}{x^2+px+q}+\frac{A_2x+B_2}{(x^2+px+q)^2}+\ldots+\frac{A_sx+B_s}{(x^2+px+q)^s}$$

mos keladi.

To'g'ri ratsional kasrning eng sodda ratsional kasrlar yig'indisiga yoyilmasida A_i, B_i koeffitsientlarni aniqlash uchun turli xil usullar mavjud. Ulardan noma'lum koeffitsientlar usuli bilan misollarda tanishamiz.

10-misol. $\int\frac{2x^2+41x-91}{(x-1)(x+3)(x-4)}dx$ topilsin.

Yechish. 34.1-teoremadan foydalanib integral ostidagi to'g'ri kasrni eng sodda ratsional kasrga yoyamiz:

$$\frac{2x^2+41x-91}{(x-1)(x+3)(x-4)}=\frac{A_1}{x-1}+\frac{A_2}{x+3}+\frac{A_3}{x-4} \qquad (34.4)$$

Bu yerdagi A_1, A_2, A_3 koeffitsientlarni topish uchun so'nggi tenglikning o'ng tomonidagi yig'indini umumiy maxrajga keltiramiz

$$\frac{2x^2+41x-91}{(x-1)(x+3)(x-4)}=\frac{A_1(x+3)(x-4)+A_2(x-1)(x-4)+A_3(x-1)(x+3)}{(x-1)(x+3)(x-4)}$$

Tenglikning har ikkala tomonidagi kasrlarning maxrajlarini tengligidan ularning suratlarini ham tengligi kelib chiqadi, ya'ni

$2x^2+41x-91=A_1(x^2-x-12)+A_2(x^2-5x+4)+A_3(x^2+2x-3)$ yoki

$2x^2+41x-91=(A_1+A_2+A_3)x^2+(-A_1-5A_2+2A_3)x+(-12A_1+4A_2-3A_3)$.

Ikkita ko'phad aynan teng bo'lishi uchun bir xil darajali x lar oldidagi koeffitsientlar teng bo'lishi kerak. Mos koeffitsientlarni o'zaro tenglashtirib

$$\begin{cases} A_1 + A_2 + A_3 = 2, (x^2 \ \ oldidagi) \\ -A_1 - 5A_2 + 2A_3 = 41, (x \ \ oldidagi) \\ -12A_1 + 4A_2 - 3A_3 = -91 (ozod \ \ sonlar) \end{cases}$$

uchta A_1 , A_2 , A_3 noma'lumli uchta tenglamalar sistemasini hosil qilamiz. Shu sistemani yechamiz.

Sistemaning birinchi tenglamasini ikkinchi tenglamasiga va birinchi tenglamani 12 ga ko'paytirib uchinchi tenglamasiga qo'shsak

$$\begin{cases} -4A_2 + 3A_3 = 43, \\ 16A_2 + 9A_3 = -67 \end{cases}$$

sistema hosil bo'ladi. Bu sistemaning birinchi tenglamasini 4 ga ko'paytirib, ikkinchi tenglamaga qo'shsak $21A_3 = 105$ tenglik hosil bo'lib undan $A_3 = 5$ kelib chiqadi. $A_3 = 5$ qiymatni oxirgi sistemaning birinchi tenglamasiga qo'yib A_2 ni topamiz: *oldidagi*

$$-4A_2 + 3 \cdot 5 = 43; \qquad -4A_2 = 28; \qquad A_2 = -7.$$

$A_3 = 5$, $A_2 = -7$ qiymatlarni $A_1 + A_2 + A_3 = 2$ tenglikka qo'yib A_1 ni topamiz:

$$A_1 - 7 + 5 = 2, \ A_1 = 4.$$

A_1, A_2 va A_3 larning topilgan qiymatlarini (34.4) ga qo'ysak

$$\frac{2x^2 + 41x - 91}{(x-1)(x+3)(x-4)} = \frac{4}{x-1} - \frac{7}{x+3} + \frac{5}{x-4}$$

yoyilma hosil bo'ladi. Uni integrallab berilgan integralni topamiz:

$$\int \frac{2x^2 + 41x - 91}{(x-1)(x+3)(x-4)} dx = \int \frac{4dx}{x-1} - \int \frac{7dx}{x+3} + \int \frac{5dx}{x-4} =$$

$$= 4\ln|x-1| - 7\ln|x+3| + 5\ln|x-4| + c = \ln(x-1)^4 + \ln|x-4|^5 - \ln|x+3|^7 + C$$

$$= \ln\left|\frac{(x-1)^4 (x-4)^5}{(x+3)^7}\right| + C.$$

11-misol. $\int \frac{(3x^2 + 8)dx}{x^3 + 4x^2 + 4x}$ topilsin.

Yechish. Integral ostidagi to‘g‘ri kasrning maxraji

$$x^3+4x^2+4x=x(x^2+4x+4)=x(x+2)^2$$

bo‘lgani uchun 34.1 –teoremaga ko‘ra

$$\frac{3x^2+8}{x^3+4x^2+4x}=\frac{A}{x}+\frac{B}{(x+2)}+\frac{C}{(x+2)^2}$$

bo‘ladi. Buni

$$\frac{3x^2+8}{x^3+4x^2+4x}=\frac{A(x+2)^2+Bx(x+2)+Cx}{x(x+2)^2}$$

ko‘rinishida yozib

$$3x^2+8=A(x^2+4x+4)+B(x^2+2x)+Cx=(A+B)x^2+(4A+2B+C)x+4A$$

tenglikka kelamiz. Ikki ko‘phadni tengligidan foydalanib

$$\begin{cases} A+B=3, \\ 4A+2B+C=0, \\ 4A=8 \end{cases}$$

sistemani hosil qilamiz va uni yechib

$$A=2, \quad B=1, \quad C=-10$$

bo‘lishini topamiz. Demak

$$\frac{3x^2+8}{x^3+4x^2+4x}=\frac{2}{x}+\frac{1}{x+2}+\frac{-10}{(x+2)^2}.$$

Buni integrallab berilgan integralni topamiz:

$$\int\frac{3x^2+8}{x^3+4x^2+4x}dx=2\int\frac{dx}{x}+\int\frac{dx}{x+2}-10\cdot\int\frac{dx}{(x+2)^2}=$$

$$=2\ln|x|+\ln|x+2|-10\cdot\left(-\frac{1}{x+2}\right)+C=\ln\left|x^2(x+2)\right|+\frac{10}{x+2}+C.$$

12-misol. $\int\frac{x^2-5x+9}{(x-1)^2(x^2+2x+2)}dx$ topilsin.

Yechish: 34.1- teoremaga ko‘ra

$$\frac{x^2-5x+9}{(x-1)^2(x^2+2x+2)}=\frac{A_1}{x-2}+\frac{A_2}{(x-1)^2}+\frac{Mx+N}{x^2+2x+2} \qquad (34.5)$$

yoyilmaga ega bo'lamiz, bunda A_1, A_2, M, N hozircha noma'lum sonlar. (34.5) tenglikning o'ng tomonini umumiy maxrajga keltiramiz:

$$\frac{x^2-5x+9}{(x-1)^2(x^2+2x+2)}=\frac{A_1(x-1)(x^2+2x+2)+A_2(x^2+2x+2)+(Mx+N)(x-1)^2}{(x-1)^2(x^2+2x+2)}$$

Bu ayniyatning maxrajlari teng bo'lgani uchun ularning suratlari ham o'zaro teng bo'ladi.

$$x^2-5x+9=A_1(x-1)(x^2+2x+2)+A_2(x^2+2x+2)+(Mx+N)(x^2-2x+1)$$

yoki

$$x^2-5x+9=A_1(x^3+x^2-2)+A_2(x^2+2x+2)+M(x^3-2x^2+x)+N(x^2-2x+1).$$

Bu yerdagi qavslarni ochib ko'phadni x ning darajalarini kamayishi tartibida joylashtirsak

$$x^2-5x+9=(A+M)x^3+(A_1+A_2-2M+N)x^2+(2A_2+M-2N)x+$$
$$+(-2A_1+2A_2+N)$$

bo'ladi.

Tenglikning har ikkala tomonidagi bir xil darajali x lar oldidagi koeffitsientlarni tenglashtirib quyidagi tenglamalar sistemasini hosil qilamiz:

$$\begin{cases} A_1+M=0,(x^3 \ \text{oldidagi}) \\ A_1+A_2-2M+N=1,(x^2 \ \text{oldidagi}) \\ 2A_2+M-2N=-5,(x \ \text{oldidagi}) \\ -2A_1+2A_2+N=9.(\text{ozod sonlar}). \end{cases}$$

Shu sistemani yechib A_1, A_2, M, N larni topamiz. Birinchi tenglamadan $A=-M$ ni topib uni boshqa tenglamalarga qo'ysak

$$\begin{cases} A_1 = -M, \\ A_2 - 3M + N = 1, \\ 2A_2 + M - 2N = -5, \\ 2A_2 + 2M + N = 9 \end{cases}$$

sistemaga ega bo‘lamiz. Ikkinchi tenglamani *-2* ga ko‘paytirib uchinchi va to‘rtinchi tenglamalarga qo‘shsak

$$\begin{cases} A_1 = -M, \\ A_2 - 3M + N = 1, \\ 7M - 4N = -7, \\ 8M - N = 7 \end{cases}$$

sistema kelib chiqadi. Bu sistemani oxirgi tenglamasini -4ga ko‘paytirib uchinchi tenglamasiga qo‘shsak $-25M=-35$ *va* bundan $M=\frac{7}{5}$ kelib chiqadi. Buni oxirgi sistemaning to‘rtinchi tenglamasi $8M-N=7$ ga qo‘yib N ni aniqlaymiz:

$$N=8M-7=8\cdot\frac{7}{5}-7=\frac{56-35}{5}=\frac{21}{5}.$$

M va N ning topilgan qiymatlarini oxirgi sistemaning $A_2-3M+N=1$ tenglamasiga qo‘yib A_2 ni topamiz:

$$A_2 = 3M+1-N = 3\cdot\frac{7}{5}+1-\frac{21}{5} = 1.$$

Sistemaning birinchi tenglamasidan $A_1=-M=-\frac{7}{5}$ hosil bo‘ladi. Shunday qilib

$$A_1=-\frac{7}{5},\ A_2=1,\ M=\frac{7}{5},\ N=\frac{21}{5}$$

yechimga ega bo‘lamiz. Ushbu qiymatlarni (34.5) ga qo‘yib

$$\frac{x^2-5x+9}{(x-1)^2(x^2+2x+2)} = -\frac{7}{5(x-1)}+\frac{1}{(x-1)^2}+\frac{7x+21}{5(x^2+2x+2)}$$

yoyilmani hosil qilamiz. Buni integrallab berilgan integralni topamiz:

$$\int \frac{x^2-5x+9}{(x-1)^2(x^2+2x+2)}dx =$$

$$= -\frac{7}{5}\int\frac{dx}{x-1} + \int\frac{dx}{(x-1)^2} + \frac{7}{5}\int\frac{x+3}{x^2+2x+2}dx = -\frac{7}{5}\ln|x-1| - \frac{1}{x-1} +$$

$$+\frac{7}{5}\cdot\frac{1}{2}\int\frac{2x+2+4}{x^2+2x+2}dx = -\frac{7}{5}\ln|x-1| - \frac{1}{x-1} + \frac{7}{10}\int\frac{(2x+2)dx}{x^2+2x+2} + \frac{14}{5}\int\frac{dx}{x^2+2x+2} = -\frac{7}{5}\ln|x-1| -$$

$$\frac{1}{x-1} + \frac{7}{10}\int\frac{(x^2+2x+2)'dx}{x^2+2x+2} + \frac{14}{5}\int\frac{d(x+1)}{1+(x+1)^2} = -\frac{7}{5}\ln|x-1| - \frac{1}{x-1} + \frac{7}{10}\ln(x^2+2x+2) +$$

$$+\frac{14}{5}\operatorname{arctg}(x+1) + C$$

13-misol. $\int\frac{x^6+2x^4+2x^2-1}{x^5+2x^3+x}dx$ integral topilsin.

Yechish. Integral ostidagi funksiya kasr-ratsional funksiya bo‘lib, u noto‘g‘ri kasrdir. Bu kasrning suratini uning maxrajiga bo‘lib kasrning butun qismini ajratamiz:

$$\begin{array}{r|l} -\,x^6+2x^4+2x^2-1 & x^5+2x^3+x \\ \cline{2-2} x^6+2x^4+x^2 & x \\ \hline x^2-1 & \end{array}$$

Kasrning maxraji $x^5+2x^3+x=x(x^4+2x^2+1)=x(x^2+1)^2$ ko‘rinishdagi ko‘pytuvchilarga ajralishini hisobga olsak

$$\frac{x^6+2x^4+2x^2-1}{x^5+2x^3+x} = x + \frac{x^2-1}{x(x^2+1)^2}$$

tenglikka ega bo‘lamiz.

Endi

$$\frac{x^2-1}{x(x^2+1)^2}$$

to‘g‘ri kasrni 34.1-teoremadan foydalanib eng sodda ratsional kasrlarga yoyamiz:

$$\frac{x^2-1}{x(x^2+1)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+1}+\frac{Dx+E}{(x^2+1)^2} \qquad (34.6),$$

$$x^2-1=A(x^2+1)^2+(Bx+C)x(x^2+1)+(Dx+E)x=A(x^4+2x^2+1)+$$
$$+Bx^4+Cx^3+Bx^2+Cx+$$
$$+Dx^2+Ex=(A+B)x^4+Cx^3+(2A+B+D)x^2+(C+E)x+A.$$

Bu yerdagi bir xil darajali x lar oldidagi koefitsiyentlarni tenglashtirib

$$\begin{cases} A+B=0, \\ C=0, \\ 2A+B+D=1, \\ C+E=0, \\ A=-1 \end{cases}$$

sistemaga ega bo‘lamiz. Sistemani yechib $A=-1,\ B=1,\ C=0,\ D=2,\ E=0$ ekanini topamiz. Noma’lum koeffitsiyentlarning topilgan qiymatlari (34.6) ga qo‘yib

$$\frac{x^2-1}{x(x^2+1)^2}=-\frac{1}{x}+\frac{x}{x^2+1}+\frac{2x}{(x^2+1)^2}$$

tenglikni hosil qilamiz.

Shunday qilib

$$\frac{x^6+2x^4+2x^2-1}{x^5+2x^3+x}=x-\frac{1}{x}+\frac{x}{x^2+1}+\frac{2x}{(x^2+1)^2}.$$

Demak,

$$\int\frac{x^6+2x^4+2x^2-1}{x^5+2x^3+x}dx=\int xdx-\int\frac{dx}{x}+\int\frac{xdx}{x^2+1}+\int\frac{2xdx}{(x^2+1)^2}=\frac{x^2}{2}-\ln|x|+\frac{1}{2}\ln(x^2+1)+$$

$$+\int\frac{d(x^2+1)}{(x^2+1)^2}=\frac{x^2}{2}-\ln|x|+\frac{1}{2}\ln(x^2+1)-\frac{1}{x^2+1}+C$$

bo‘ladi.

Eslatma. Ba'zan ratsional kasrni eng sodda ratsional kasrga yoyishda aniqmas koeffitsiyentlar usuliga murojaat qilmasdan uni sun'iy ravishda eng sodda kasrlarga yoyish ham mumkin.

Buni bir necha misollar yordamida ko'rsatamiz.

14-misol. $\int \frac{dx}{(x+a)(x+b)}$ *(a≠b)* topilsin.

Yechish.

$$\frac{1}{(x+a)(x+b)} = \frac{1}{a-b}\left(\frac{1}{x+b} - \frac{1}{x+a}\right)$$

ayniyat to'g'ri ekanligini ko'rsatish qiyin emas. Demak,

$$\int \frac{dx}{(x+a)(x+b)} = \frac{1}{a-b}\left(\int \frac{dx}{x+b} - \int \frac{dx}{x+a}\right) =$$

$$= \frac{1}{a-b}(\ln|x+b| - \ln|x+a|) + C = \frac{1}{a-b}\ln\left|\frac{x+b}{x+a}\right| + C.$$

Quyidagi integrallar ham xuddi shunday topiladi.

15-misol.

$$\int \frac{dx}{(x^2+4)(x^2+1)} = \frac{1}{3}\int\left(\frac{1}{x^2+1} - \frac{1}{x^2+4}\right)dx = \frac{1}{3}\text{arctg}x - \frac{1}{6}\text{arctg}\frac{x}{2} + C.$$

16-misol.

$$\int \frac{dx}{x^4-x^2} = \int \frac{dx}{(x^2-1)x^2} = \int\left(\frac{1}{x^2-1} - \frac{1}{x^2}\right)dx = \int \frac{dx}{x^2-1} - \int \frac{dx}{x^2} =$$

$$= \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + \frac{1}{x} + C$$

17-misol.

$$\int \frac{dx}{x^3+3x} = \int \frac{dx}{x(x^2+3)} = \frac{1}{3}\int \frac{x^2+3-x^2}{x(x^2+3)}dx = \frac{1}{3}\int\left[\frac{x^2+3}{x(x^2+3)} - \frac{x^2}{x(x^2+3)}\right]dx =$$

$$= \frac{1}{3}\int \frac{dx}{x} - \frac{1}{3}\int \frac{xdx}{x^2+3} = \frac{1}{3}\ln|x| - \frac{1}{6}\ln(x^2+3) + C.$$

18-misol.

$$\int\frac{dx}{x^4-x^2-6}=\int\frac{dx}{(x^2-3)(x^2+2)}=\frac{1}{5}\int(\frac{1}{x^2-3}-\frac{1}{x^2+2})dx=$$

$$=\frac{1}{5}\int\frac{dx}{x^2-(\sqrt{3})^2}-\frac{1}{5}\int\frac{dx}{(\sqrt{2})^2+x^2}=\frac{1}{10\sqrt{3}}\ln\left|\frac{x-\sqrt{3}}{x+\sqrt{3}}\right|-\frac{1}{5\sqrt{2}}\text{arctg}\frac{x}{\sqrt{2}}+C.$$

19-misol.

$$\int\frac{dx}{x^4-16}=\int\frac{dx}{(x^2-4)(x^2+4)}=\frac{1}{8}\int(\frac{1}{x^2-4}-\frac{1}{x^2+4})dx=$$

$$\frac{1}{32}\ln\left|\frac{x-2}{x+2}\right|-\frac{1}{16}\text{arctg}\frac{x}{2}+C.$$

20-misol.

$$\int\frac{dx}{(x^2+2x+2)(x^2+2x+5)}=\int\frac{dx}{\left[(x+1)^2+1\right]\cdot\left[(x+1)^2+4\right]}=$$

$$\frac{1}{3}\int\left[\frac{1}{(x+1)^2+1}-\frac{1}{(x+1)^2+2^2}\right]dx=\frac{1}{3}\text{arctg}(x+1)-\frac{1}{6}\text{arctg}\frac{x+1}{2}+C.$$

21-misol.

$$\int\frac{x^2dx}{1-x^4}=\frac{1}{2}\int(\frac{1}{1-x^2}-\frac{1}{1+x^2})dx=\frac{1}{4}\ln\left|\frac{1+x}{1-x}\right|-\frac{1}{2}\text{arctg}x+C.$$

22-misol.

$$\int\frac{dx}{x^8+x^6}=\int\frac{dx}{x^6(x^2+1)}=-\int\frac{x^4-1-x^4}{x^6(x^2+1)}dx=-\int\frac{(x^2-1)(x^2+1)-x^4}{x^6(x^2+1)})dx=$$

$$\int\left[\frac{x^2-1}{x^6}-\frac{1}{x^2(x^2+1)}\right]dx=-\int(\frac{1}{x^4}-\frac{1}{x^6}-\frac{1}{x^2}+\frac{1}{1+x^2})dx=$$

$$-(\frac{x^{-4+1}}{-4+1}-\frac{x^{-6+1}}{-6+1}+\frac{1}{x}+\text{arctg}x)+C=\frac{1}{3x^3}-\frac{1}{5x^5}-\frac{1}{x}-\text{arctg}x+C.$$

23-misol.

$$\int\frac{x^2dx}{(x-1)^{12}}\left|\begin{matrix}x-1=t, x=1+t\\ dx=dt\end{matrix}\right|=\int\frac{(1+t)^2dt}{t^{12}}=$$

$$=\int\frac{1+2t+t^2}{t^{12}}dt=\int(\frac{1}{t^{12}}+\frac{2}{t^{11}}+\frac{1}{t^{10}})dt=\frac{t^{-12+1}}{-12+1}+$$

$$+2\cdot\frac{t^{-11+1}}{-11+1}+\frac{t^{-10+1}}{-10+1}+C=-\frac{1}{11t^{11}}-\frac{1}{10t^{10}}-\frac{1}{9t^9}+C=$$

$$=-\frac{1}{11(x-1)^{11}}-\frac{1}{10(x-1)^{10}}-\frac{1}{9(x-1)^9}+C.$$

34.4. Ratsional funksiyani integrallash

Butun (ko‘phad) va kasr-ratsional funksiyalar odatda ratsional funksiya deb yuritiladi. Agar ratsional funksiya butun ratsional funksiya (ko‘phad) bo‘lsa, uni integrali osonlikcha topilishini ko‘rdik. Agar ratsional funksiya kasr- ratsional funksiya bo‘lsa uni integrallash quyidagi qoidaga asoslanib amalga oshiriladi.

1. Ratsional kasr noto‘g‘ri bo‘lsa, kasrning suratini maxrajiga bo‘lib uni ko‘phad bilan to‘g‘ri kasrning yig‘indisi ko‘rinishida tasvirlanadi.

Bu bilan noto‘g‘ri ratsional kasrni integrallash ko‘phad bilan to‘g‘ri ratsional kasrni integrallashga keltiriladi.

2.To‘g‘ri ratsional kasrning maxraji ko‘paytuvchilarga ajratiladi.

3.To‘g‘ri ratsional kasr 34.1-teoremadan foydalanib eng sodda ratsional kasrlarning yig‘indisi ko‘rinishida tasvirlanadi. Bu bilan to‘g‘ri ratsional kasrni integrllash eng sodda ratsional kasrlarni integrllashga keltiriladi.

Xulosa. Har qanday ratsional funksiyani integrallasa bo‘lar ekan, ya‘ni ratsional funksiyaning integrali elementar funksiya bo‘lar ekan.

O‘z-o‘zini tekshirish uchun savollar

1. Kvadrat uchhad qatnashgan funksiya qanday integrallanadi?
2. I-,II-, III- va IV-tur eng sodda ratsional kasr deb nimaga aytiladi va ular qanday integrallanadi?
3. Rekkurent formula deb ataluvchi formulani isbotlang.
4. Qanaqa turdagi eng sodda ratsional kasrlarni integrallari mavjud?
5. $\frac{x+1}{x^2-5x+6}$ kasr eng sodda ratsional kasrmi?
6. $\frac{3}{x^2+4x+4}$ kasr qaysi turdagi eng sodda ratsional kasr?
7. $\frac{7}{(x^2+6x+9)^{10}}$ kasr qaysi turdagi eng sodda ratsional kasr?
8. $\frac{x^2-1}{x^4+1}$ kasr qaysi turdagi eng sodda ratsional kasr.
9. Butun ratsional funksiya nima?
10. Kasr- ratsional funksiya nima?
11. Ratsional kasr qachon to‘g‘ri deyiladi?
12. Ratsional kasr qachon noto‘g‘ri deyiladi?
13. To‘g‘ri ratsional kasr qanday qilib eng sodda ratsional kasrlarning yig‘indisi ko‘rinishida tasvirlanadi?
14. Noma‘lum koeffitsientlar usuli nimadan iborat?
15. Ratsional funksiyani integrallash qanday tartibda amalga oshiriladi?
16. Har qanday ratsional funksiyani integrali mavjudmi?
17. Noto‘g‘ri ratsional kasr qanday integrallanadi?

Mustaqil yechish uchun mashqlar

Integrallar topilsin:

1. $\int \frac{dx}{x^2+3x+1}$. Javob: $\frac{1}{\sqrt{5}}\ln\left|\frac{2x+3-\sqrt{5}}{2x+3+\sqrt{5}}\right|+C.$

2. $\int \frac{dx}{2x^2-2x+1}$. Javob: $\operatorname{arctg}(2x-1)+C$.

3. $\int \frac{3x-1}{x^2-x+1}dx$. Javob: $\frac{3}{2}\ln(x^2-x+1)+\frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2x-1}{\sqrt{3}}+C$.

4. $\int \frac{dx}{\sqrt{2-3x-4x^2}}$. Javob: $\frac{1}{2}\arcsin\frac{8x+3}{\sqrt{41}}+C$.

5. $\int \frac{(x+3)dx}{\sqrt{3+4x-4x^2}}$. Javob: $-\frac{1}{4}\sqrt{3+4x-4x^2}+\frac{7}{4}\arcsin\frac{2x-1}{2}+C$.

6. $\int \frac{dx}{x^2+2x+5}$. Javob: $\frac{1}{2}\text{arctg}\frac{x+1}{2}+C$.

7. $\int \frac{dx}{2x^2-2x+1}$. Javob: $\text{arctg}(2x-1)+C$.

8. $\int \frac{(8x+6)dx}{4x^2+6x-3}$. Javob: $\ln\left|4x^2+6õ-3\right|+C$.

9. $\int \frac{3x-1}{x^2-x+1}dx$. Javob : $\frac{3}{2}\ln\left(x^2-x+1\right)+\frac{1}{\sqrt{3}}\text{arctg}\frac{2x-1}{\sqrt{3}}+C$.

10. $\int \frac{x-1}{(x^2+2x+3)^2}dx$. Javob: $-\frac{x+2}{2\left(x^2+2x+3\right)}-\frac{\sqrt{2}}{4}\text{arctg}\frac{x+1}{\sqrt{2}}+C$.

11. $\int \frac{(3x+5)dx}{(x^2+2x+2)^2}$. Javob: $\frac{2õ-1}{2\left(x^2+2x+2\right)}+\text{arctg}(x+1)+C$.

12. $\int \frac{(2x-3)dx}{(x^2-4x+8)^3}$. Javob: $\frac{3x^3-18x^2+56x-128}{128\left(x^2-4x+8\right)^2}+\frac{3}{256}\text{arctg}\frac{x-2}{2}+C$.

13. $\int \frac{x^2-7x-20}{(x+1)(x-2)(x+3)}dx$. Javob: $\ln\frac{(x+1)^2|x+3|}{(x-2)^2}+C$.

14. $\int \frac{3x^3+9x^2+8x-2}{(x+5)(x+1)^3}dx$. Javob: $\frac{1}{2(x+1)^2}+3\ln|x+5|+C$.

15. $\int \frac{(x^3-3x)dx}{(x-1)^2(x^2+1)}dx$. Javob: $\frac{1}{x-1}+\ln|x-1|+2\text{arctg}x+C$.

16. $\int \frac{x^3+2}{x^3-2x^2+2x)}dx$. Javob: $x+\ln\left(|x|\cdot\sqrt{x^2-2x+2}\right)+\text{arctg}\,(x-1)+C$.

17. $\int \frac{2x^2+2x-4}{(x^2+4)(x^2+2x+2)}dx$.

Javob: $\frac{1}{2}\ln\frac{x^2+4}{x^2+2x+2}+\mathrm{arctg}\frac{x}{2}-\mathrm{arctg}(x+1)+C$.

18. $\int \frac{3x^2-10x+12}{x^4+13x^2+36}dx$. Javob: $\ln\frac{x^2+9}{x^2+4}+\mathrm{arctg}\frac{x}{3}+C$.

19. $\int \frac{dx}{(4-x)(x+2)}$. Javob: $\frac{1}{6}\ln\left|\frac{x+2}{x-4}\right|+C$.

20. $\int \frac{dx}{(x^2-4)(x^2+1)}$. Javob: $\frac{1}{4}\ln\left|\frac{x-2}{x+2}\right|-\mathrm{arctg}x+C$.

21. $\int \frac{x^2dx}{(1+x^2)(4+x^2)}$. Javob: $\frac{2}{3}\mathrm{arctg}\frac{x}{2}-\frac{1}{3}\mathrm{arctg}x+C$.

22. $\int \frac{x^2dx}{(1-x^2)(4+x^2)}dx$. Javob: $-\frac{8}{5}\ln\left|\frac{x-1}{x+1}\right|-\frac{2}{5}\mathrm{arctg}\frac{x}{2}+C$.

35. TRIGONOMETRIK FUNKSIYALAR QATNASHGAN IFODALARNI INTEGRALLASH

35.1. Ikki o'zgaruvchining ratsional funksiyasi

Ikkita u va v o'zgaruvchilarning Au^nv^m ko'rinishdagi ko'paytmalarining yig'indisidan tuzilgan ifoda **u va v o'zgaruvchilarning ko'phadi** deyiladi, bunda A o'zgarmas haqiqiy son n va m manfiy bo'lmagan butun sonlar.

Masalan, $u^5+3u^2v+4uv^2-2uv+3$ va $8u^3+4v$ ifodalar u va v ning ko'phadidir.

u va v ning ikkita ko'phadlarini nisbati u va v ning **ratsional funksiyasi** yoki ularning **ratsional ifodasi** deb ataladi. u va v ning ratsional funksiyasi $R(u\,;\,v)$ kabi belgilanadi.

Masalan, $\frac{u^2+2uv-1}{u^3+4uv^2+v^3}, \frac{3u^2-2uv+1}{8uv}, \frac{5}{u^2+v^2}$

kasrlar u va v ning ratsional funksiyalaridir. u va v ning ratsional funksiyalarini yig'indisi, ayirmasi, ko'paytmasi, bo'linmasi ratsional funksiyaning ratsional funksiyasi ham shu o'zgaruvchilarning ratsional funksiyasi bo'ladi.

u va v o'zgaruvchilarning har biri o'z navbatida x o'zgaruvchining $u=\varphi(x), v=\psi(x)$ funksiyalari bo'lganda $R(\varphi(x),\psi(x))$ funksiya $\varphi(x)$ va $\psi(x)$ funksiyalarning ratsional funksiyasi bo'ladi.

Masalan, $f(x)=\frac{x+3\sqrt{x^2+1}+1}{\sqrt{x}+x+x^2+1}$ funksiya $u=\sqrt{x}, v=\sqrt{x^2+1}$ larning ratsional funksiyasi bo'ladi, chunki

$$R(u,v)=\frac{u^2+3v+1}{u+u^2+v^2}.$$

Shunga o'xshash $\frac{\sin^2 x+\cos^3 x}{\sin^3 x+2\cos^2 x}$ ifoda $\sin x$ va $\cos x$ larning ratsional funksiyasi.

Uch va undan ortiq o'zgaruvchilarning ratsional funksiyasi ham xuddi ikki o'zgaruvchining ratsional funksiyasi kabi aniqlanadi. Bu tushunchalardan kelgusida trigonometrik funksiyalar qatnashgan ifodalarni hamda ba'zi bir irratsional funksiyalarni integrallashda foydalanamiz.

35.2. $\int \sin^n x \cdot \cos^m x dx$ ko'rinishdagi integral

Bu integral *m* va *n* (butun sonlar) ga bog'liq ravishda har xil almashtirishlar olish bilan ratsional funksiyaning integraliga keltiriladi ya'ni ratsionallashtiriladi.

a) $n>0$ va toq son bo'lsin. U holda

$$\cos x = z\,,\quad \sin x dx = -dz$$

almashtirish yordamida berilgan integral ratsionallashtiriladi.

b) $m>0$ va toq son bo'lsin. U holda berilgan integral

$$\sin x = z,\quad \cos x dx = dz$$

almashtirish yordamida ratsionallashtiriladi.

c) *n* va *m* darajalar juft va nomanfiy bo'lsin.

U holda

$$\sin^2 x = \frac{1-\cos 2x}{2}, \cos^2 x = \frac{1+\cos 2x}{2}$$

darajani pasaytirish formulalaridan foydalanib berilgan integralni $\cos 2x$ ko'phadining integraliga ega bo'lamiz. $\cos 2x$ ning toq darajalari ishtirok etgan integrallar *b)* bandga asosan topiladi. $\cos 2x$ ning juft darajalari ishtirok etgan integrallarni topish uchun yana

$$\cos^2 2x = \frac{1+\cos 4x}{2}$$

darajani pasaytirish formulasidan foydalanamiz. Bu jarayonni davom ettirib oxiri

$$\int \cos kx dx = \frac{1}{k}\sin kx + C$$

ga ega bo'lamiz.

d) n va m juft sonlar bo'lib ulardan kamida bittasi manfiy bo'lsin. U holda

$$\mathrm{tg}x = z, x = \mathrm{arctg}z, dx = \frac{dz}{1+z^2}, \cos^2 x = \frac{1}{1+tg^2x} = \frac{1}{1+z^2},$$

$$\sin^2 x = \frac{tg^2x}{1+tg^2x} = \frac{z^2}{1+z^2}$$

almashtirish yordamida berilgan integral ratsionallashtiriladi.

1-misol. $\int \frac{cos^3 x}{\sin^2 x} dx$ hisoblansin.

Yechish. $m=3$ toq son.

$$\int \frac{\cos^3 x}{\sin^2 x} dx = \int \frac{\cos^2 x \cdot \cos x dx}{\sin^2 x} = \int \frac{(1-\sin^2 x)\cos x dx}{\sin^2 x}\left|\begin{matrix}\sin x = z \\ \cos x dx = dz\end{matrix}\right| =$$

$$= \int \frac{(1-z^2)dz}{z^2} = \int\left(\frac{1}{z^2}-1\right)dz = -\frac{1}{z} - z + C = \frac{1}{\sin x} - \sin x + C.$$

2-misol. $\int \sin^4 x cos^2 x dx$ topilsin.

Yechish. $n=4,\ m=2$ juft musbat sonlar.

$$\int \sin^4 x\cos^2 x dx = \int (\sin^2 x)^2 \cos^2 x dx =$$

$$= \int \left(\frac{1-\cos 2x}{2}\right)^2 \frac{1+\cos 2x}{2} dx =$$

$$= \frac{1}{8}\int (1-\cos 2x)(1-\cos 2x)(1+\cos 2x)dx =$$

$$= \frac{1}{8}\int (1-\cos 2x)(1-\cos^2 2x)dx =$$

$$= \int \frac{1}{8}(1-\cos 2x)\sin^2 2x dx = \frac{1}{8}\int \sin^2 2x dx -$$

$$-\frac{1}{8}\int \sin^2 2x\cos 2x dx = \frac{1}{8}\int \frac{1-\cos 4x}{2}dx -$$

$$-\frac{1}{8}\cdot\frac{1}{2}\int \sin^2 2x(\sin 2x)'dx=\frac{1}{16}\int (1-cos4x)dx-$$

$$-\frac{1}{16}\int \sin^2 2xd(\sin 2x)=\frac{1}{16}x-\frac{\sin 4x}{64}-\frac{\sin^3 2x}{48}+C.$$

3-misol. $\int\frac{\sin^2 x}{\cos^6 x}dx$ topilsin.

Yechish. $n=2, m=-6$ juft sonlar va $m<O$.

$$\int\frac{\sin^2 x}{\cos^6 x}dx=\int\frac{\sin^2 x}{\cos^2 x}\cdot\frac{1}{\cos^2 x}\cdot\frac{dx}{\cos^2 x}=\int(tgx)^2\cdot(1+tg^2x)\frac{dx}{\cos^2 x}\left|\begin{matrix}tgx=z\\ dz=\dfrac{dx}{\cos^2 x}\end{matrix}\right|=$$

$$\int z^2(1+z^2)dz=\int(z^2+z^4)dz=\frac{z^3}{3}+\frac{z^5}{5}+C=\frac{tg^3x}{3}+\frac{tg^5x}{5}+C.$$

35.3. $\int$R(sinx,cosx)dx ko'rinishdagi integral

Ushbu $\int R(\sin x,\cos x)dx$ integralni qaraymiz.

Bu integral

$$tg\frac{x}{2}=z,\frac{x}{2}=\text{arctgz}, x=2\text{arctgz}, dx=\frac{2dz}{1+z^2}$$

almashtirish yordamida ratsionallashtiriladi.

Haqiqatan, $\cos^2\frac{x}{2}+\sin^2\frac{x}{2}=1$ ekanini e'tiborga olib

$$\sin x=2\sin\frac{x}{2}\cdot\cos\frac{x}{2}=\frac{2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}{\sin^2\frac{x}{2}+\cos^2\frac{x}{2}}=\frac{2tg\frac{x}{2}}{tg^2\frac{x}{2}+1}=\frac{2z}{1+z^2},$$

$$\cos x=\cos^2\frac{x}{2}-\sin^2\frac{x}{2}=\frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}}=\frac{1-tg^2\frac{x}{2}}{1+tg^2\frac{x}{2}}=\frac{1-z^2}{1+z^2},$$

formulalarga ega bo'lamiz.

Shuning uchun

$$\int R(\sin x, \cos x)dx = \int R\left(\frac{2z}{1+z^2}, \frac{1-z^2}{1+z^2}\right)\frac{2dz}{1+z^2} = \int R_1(z)dz$$

ratsional funksiyaning integraliga ega bo'lamiz. $tg\frac{x}{2} = z$ almashtirish R(sinx,cosx) ko'rinishdagi har qanday funksiyani integrallashga imkon beradi, shuning uchun u **universal** trigonometrik almashtirish deyiladi. Lekin amaliyotda bu almashtirish ko'pincha ancha murakkab ratsional funksiyani integrallashga olib kelishi ham mumkin. Shuning uchun ba'zan undan foydalanmasdan ancha sodda o'rniga qo'yish usulidan foydalangan ma'qul.

a) Agar $R(\sin x,\cos x)$ funksiya sin$x$ ga nisbatan toq, ya'ni

$$R(-\sin x,\cos x)=-R(\sin x,\cos x)$$

bo'lsa, u holda

$$\cos x=z,\quad dz=-\sin x dx$$

almashtirish yordamida berilgan integral ratsionallashtiriladi.

b) Agar $R(\sin x,\cos x)$ funksiya *cosx* ga nisbatan toq bo'lsa, u holda

$$\sin x=z,\quad \cos x dx=dz$$

almashtirish yordamida integral ratsionallashtiriladi.

c) Agar $R(\sin x,\cos x)$ funksiya sin$x$ va cos$x$ ga nisbatan juft, ya'ni

$$R(-\sin x,-\cos x)= R(\sin x,\cos x)$$

bo'lsa, u holda integral

$$tgx = z, dx = \frac{dz}{1+z^2}$$

almashtirish yordamida ratsionallashtiriladi.

Bu holda

$$\sin^2 x = \frac{tg^2 x}{1+tg^2 x} = \frac{z^2}{1+z^2}, \cos^2 x = \frac{1}{1+tg^2 x} = \frac{1}{1+z^2}$$

tengliklardan foydalaniladi.

4-misol $\int \frac{dx}{\sin x}$ topilsin.

Yechish.

$$\int \frac{dx}{\sin x} \left| \begin{array}{l} tg\frac{x}{2} = z, dx = \frac{2dz}{1+z^2} \\ \sin x = \frac{2z}{1+z^2} \end{array} \right| = \int \frac{\frac{2dz}{1+z^2}}{\frac{2z}{1+z^2}} = \int \frac{dz}{z} = \ln|z| + C = \ln\left|tg\frac{x}{2}\right| + C.$$

5-misol $\int \frac{dx}{\cos x + 2\sin x + 3}$ topilsin.

Yechish.

$$\int \frac{dx}{\cos x + 2\sin x + 3} \left| \begin{array}{l} tg\frac{x}{2} = z, dx = \frac{2dz}{1+z^2} \\ \cos x = \frac{1-z^2}{1+z^2}, \sin x = \frac{2z}{1+z^2} \end{array} \right| = \int \frac{\frac{2dz}{1+z^2}}{\frac{1-z^2}{1+z^2} + 2\cdot\frac{2z}{1+z^2} + 3} =$$

$$\int \frac{dz}{z^2 + 2z + 2} = \int \frac{dz}{(z+1)^2 + 1} = \operatorname{arctg}(z+1) + C = \operatorname{arctg}(tg\frac{x}{2} + 1) + C.$$

6-misol. $\int \frac{\sin^3 x}{2+\cos x} dx$ topilsin.

Yechish. Integral ostidagi funksiya *sinx* ga nisbatan toq funksiya. Shuning uchun

$$\int \frac{\sin^3 x}{2+\cos x} dx = \int \frac{\sin^2 x \cdot \sin x dx}{2+\cos x} =$$

$$\int \frac{(1-\cos^2 x)\cdot \sin x dx}{2+\cos x} \left| \begin{array}{l} \cos x = z \\ \sin x dx = -dz \end{array} \right| = \int \frac{(1-z^2)(-dz)}{2+z} =$$

$$\int \frac{(z^2-1)dz}{z+2} = \int \frac{(z^2-4+3)dz}{z+2} = \int \frac{(z-2)(z+2)}{z+2} dz + 3\int \frac{dz}{z+2} =$$

$$=\frac{z^2}{2}-2z+3\ln|z+2|+C=\frac{\cos^2 x}{2}-2\cos x++3\ln|2+\cos x|+C.$$

7-misol. $\int\frac{dx}{3-2\sin^2 x}$ topilsin.

Yechish. Integral ostidagi funksiya *sinx* va *cosx* funksiyalarga nisbatan juft funksiya. Shuning uchun

$$\int\frac{dx}{3-2\sin^2 x}\left|\begin{array}{l} tgx=z, dx=\frac{dz}{1+z^2} \\ \sin^2 x=\frac{z^2}{1+z^2}\end{array}\right|=\int\frac{\frac{dz}{1+z^2}}{3-\frac{2z^2}{1+z^2}}=\int\frac{dz}{3+z^2}=\int\frac{dz}{(\sqrt{3})^2+z^2}=$$

$$=\frac{1}{\sqrt{3}}\text{arctg}\frac{z}{\sqrt{3}}+C=\frac{1}{\sqrt{3}}\text{arctg}\left(\frac{tgx}{\sqrt{3}}\right)+C.$$

35.4. $\int\cos nx\cdot\cos mxdx\int\sin nx\cdot\cos mxdx\int\sin nx\cdot\sin mxdx(n\neq m)$ ko'rinishdagi integrallar

Maktab kursidan ma'lum

$$\cos\alpha\cos\beta=\frac{1}{2}[\cos(\alpha+\beta)+\cos(\alpha-\beta)]$$

formulaga asosan

$$\int\cos nx\cdot\cos mxdx=\frac{1}{2}\int[\cos(n+m)x+\cos(n-m)x]dx=$$

$$=\frac{1}{2}\cdot\frac{1}{n+m}\sin(n+m)x+\frac{1}{2}\cdot\frac{1}{n-m}\sin(n-m)x+C$$

tenglikka ega bo'lamiz.

Shuningdek

$$\sin\alpha\cdot\cos\beta=\frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$

formulaga binoan

$$\int \sin nx \cdot \cos mx dx = \frac{1}{2}\int [\sin(n+m) + \sin(n-m)]dx =$$

$$= -\frac{1}{2}\cdot\frac{1}{n+m}\cos(n+m)x - \frac{1}{2}\cdot\frac{1}{n-m}\cos(n-m)x + C$$

va

$$\sin\alpha \cdot \sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$$

formulaga ko'ra:

$$\int \sin nx \cdot \sin mx = \frac{1}{2}\int [\cos(n-m)x - \cos(n+m)x]dx =$$

$$= \frac{1}{2}\cdot\frac{1}{n-m}\sin(n-m)x - \frac{1}{2}\cdot\frac{1}{n+m}\sin(n+m)x + C.$$

8-misol.

$$\int \cos 6x \cdot \cos 3x dx = \frac{1}{2}\int (\cos 9x + \cos 3x)dx = \frac{1}{2}\left(\frac{1}{9}\sin 9x + \frac{1}{3}\sin 3x\right) + C =$$

$$= \frac{1}{18}\sin 9x + \frac{1}{6}\sin 3x + C$$

9-misol.

$$\int \sin 3x \cdot \cos 7x dx = \frac{1}{2}\int [\sin 10x + \sin(-4x)]dx = \frac{1}{2}\int [\sin 10x - \sin 4x)]dx =$$

$$= -\frac{1}{2}\cdot\frac{1}{10}\cos 10x + \frac{1}{2}\cdot\frac{1}{4}\cos 4x + C = -\frac{1}{20}\cos 10x + \frac{1}{8}\cos 4x + C.$$

10-misol.

$$\int \sin 6x \cdot \sin 2x dx = \frac{1}{2}\int (\cos 4x - \cos 8x)dx =$$

$$= \frac{1}{2}\left(\frac{\sin 4x}{4} - \frac{\sin 8x}{8}\right) + C = \frac{\sin 4x}{8} - \frac{\sin 8x}{16} + C$$

O'z-o'zini tekshirish uchun savollar

1. Bir o'zgaruvchining ko'phadi nima?

2. Bir o‘zgaruvchining ratsional funksiyasi nima?

3. Ikki o‘zgaruvchining ko‘phadi nima?

4. Ikki o‘zgaruvchining ratsional funksiyasi deb nimaga aytiladi?

5. $\int \sin^n x \cdot \cos^m x dx$ integral qanday topiladi, bu yerdagi n, m butun sonlar.

6. $\int R(\sin x; \cos x) dx$ integral qanday topiladi, R - ratsional funksiya.

7.Universal trigonometrik almashtirish deb qanaqa almashtirishga aytiladi?

8. $\int \cos nx \cos mx dx (n \neq m)$, $\int \sin nx \cos mx dx (n \neq m)$, $\int \sin nx \cdot \sin mx dx (n \neq m)$ integrallar qanday topiladi?

Mustaqil yechish uchun mashqlar

Integrallar topilsin

1. $\int \sin^5 x dx$. Javob: $-\cos x + \frac{2}{3}\cos^3 x - \frac{\cos^5 x}{5} + C$.

2. $\int \cos^4 x \sin^3 x dx$. Javob : $-\frac{1}{5}\cos^5 x + \frac{1}{7}\cos^7 x + C$.

3. $\int \frac{\cos^3 x}{\sin^4 x} dx$. Javob: $\frac{1}{\sin x} - \frac{1}{3} \cdot \frac{1}{\sin^3 x} + C$.

4. $\int \sin^4 x dx$. Javob: $\frac{3}{8}x - \frac{\sin 2x}{4} + \frac{\sin 4x}{32} + C$.

5. $\int \sin^4 x \cos^4 x dx$. Javob: $\frac{1}{128}(3x - \sin 4x + \frac{\sin 8x}{8}) + C$.

6. $\int tg^3 x dx$. Javob: $\frac{tg^2 x}{2} + \ln|\cos x| + C$.

7. $\int \frac{dx}{4 - 5\sin x}$. Javob: $\frac{1}{3}\ln\left|\frac{tg\frac{x}{2} - 2}{2tg\frac{x}{2} - 1}\right| + C$.

8. $\int \frac{dx}{5-3\cos x}$. Javob: $\frac{1}{2}\operatorname{arctg}\left|2tg\frac{x}{2}\right|+C$.

9. $\int \frac{\sin^2 x}{1+\cos^2 x}dx$. Javob: $\sqrt{2}\operatorname{arctg}\left(\frac{tgx}{\sqrt{2}}\right)+C$.

10. $\int \cos 4x \cos 7x dx$. Javob: $\frac{\sin 11x}{22}+\frac{\sin 3x}{6}+C$.

11. $\int \cos 2x \sin 4x dx$. Javob: $-\frac{\text{ños}\, 6x}{12}-\frac{\cos 2x}{4}+C$.

12. $\int \sin x \sin 3x dx$. Javob: $-\frac{\sin 4x}{8}+\frac{\sin 2x}{4}+C$.

36. BA'_ZI-BIR IRRATSIONAL FUNKSIYALARNI INTEGRALLASH

36.1. $\int R\left(x, x^{\frac{m}{n}}, \ldots x^{\frac{r}{s}}\right)dx$ ko'rinishdagi integrallar

Har qanday irratsional funksiyaning integrali elementar funksiya bo'lmasligi mumkin. Shuning uchun biz bu yerda va bundan keyin almashtirish yordamida ratsional funksiyaning integraliga keltiriladigan irratsional funksiyalarning integrallarini qaraymiz.

$\int R\left(x, x^{\frac{m}{n}}, \ldots x^{\frac{r}{s}}\right)dx$ ko'rinishdagi integralni qaraymiz, bunda R o'z argumuentlarining ratsional funksiyasi, $m, n, \ldots, r, s$ natural sonlar. k son $\frac{m}{n}, \ldots \frac{r}{s}$ kasrlarning umumiy maxraji bo'lganda qaralayotgan integral $x=t^k$ $dx=kt^{k-1}$ dt almashtirish yordamida ratsional funksiyaning integraliga keltiriladi (ratsionallashtiriladi).

1-misol. $\int \frac{\sqrt{x}dx}{\sqrt[4]{x^3}+1}$ integral hisoblansin.

Yechish: $\frac{1}{2} va \frac{3}{4}$ kasrlar uchun $\kappa=4$ umumiy maxraj bo'lgani uchun $x=z^4$, $dx=4z^3$ dz almashtirish yordamida berilgan integral ratsionallashtiriladi.

$$\int \frac{\sqrt{x}dx}{\sqrt[4]{x^3}+1} = \int \frac{x^{\frac{1}{2}}dx}{x^{3/4}+1} = \int \frac{(z^4)^{\frac{1}{2}}4z^3dz}{(z^4)^{\frac{3}{4}}+1} = 4\int \frac{z^2z^3dz}{z^3+1} = 4\int \frac{z^2(z^3+1-1)dz}{z^3+1} = 4\int z^2dz - 4\int \frac{z^2dz}{z^3+1} =$$

$$= 4\frac{z^3}{3} - \frac{4}{3}\int \frac{(z^3+1)'dz}{z^3+1} = \frac{4}{3}z^3 - \frac{4}{3}\ln\left|z^3+1\right| + C = \frac{4}{3}\left[x^{\frac{3}{4}} - \ln\left|x^{\frac{3}{4}}+1\right|\right] + C.$$

36.2. $\int R\left(x, \left(\frac{ax+b}{cx+d}\right)^{\frac{m}{n}}, \dots, \left(\frac{ax+b}{cx+d}\right)^{\frac{r}{s}}\right)dx$ ko'rinishdagi integrallar

Bu xildagi integrallar

$$\frac{ax+b}{cx+d} = t^k$$

almashtirish yordamida ratsionallashtiriladi, bunda k son $\frac{m}{n}, \dots, \frac{r}{s}$ kasrlarning umumiy maxraji, $m, n, \dots, r, s$ natural sonlar.

2-misol. $\int \frac{x^2 + \sqrt{1+x}}{\sqrt[3]{1+x}}dx$ integral hisoblansin.

Yechish. $\frac{1}{2}$ va $\frac{1}{3}$ kasrlar uchun 6 umumiy maxraj bo'lganligi uchun $1+x = t^6$ deymiz, u holda $\sqrt{1+x} = t^3, \sqrt[3]{1+x} = t^2$, $dx = 6t^5dt$.

Demak,

$$\int \frac{x^2 + \sqrt{1+x}}{\sqrt[3]{1+x}}dx = \int \frac{(t^6-1)^2 + t^3}{t^2} \cdot 6t^5dx = 6\int \left[t^3(t^{12} - 2t^6 + 1) + t^6\right]dt =$$

$$6\int (t^{15} - 2t^9 + t^6 + t^3)dt = 6\left(\frac{t^{16}}{16} - 2\frac{t^{10}}{10} + \frac{t^7}{7} + \frac{t^4}{4}\right) + C = 6t^4\left(\frac{t^{12}}{16} - \frac{t^6}{5} + \frac{t^3}{7} + \frac{1}{4}\right) + C =$$

$$= 6\left((1+x)^{\frac{1}{6}}\right)^4 \left(\frac{\left((1+x)^{\frac{1}{6}}\right)^{12}}{16} - \frac{1+x}{5} + \frac{\sqrt{1+x}}{7} + \frac{1}{4}\right) + C =$$

$$= 6\sqrt[3]{(1+x)^2}\left(\frac{(1+x)^2}{16} - \frac{1+x}{5} + \frac{\sqrt{1+x}}{7} + \frac{1}{4}\right) + C$$

36.3. $\int\sqrt{ax^2+bx+c}dx$ ko'rinishdagi integrallar

Bu integrallar ax^2+bx+c kvadrat uchhaddan to'liq kvadrat ajratish yo'li bilan $\int\sqrt{m^2-x^2}dx$ *yoki* $\int\sqrt{x^2\pm m^2}dx$ integrallardan biriga keltiriladi.

3-misol. $\int\sqrt{2-2x+x^2}dx$ integral hisoblansin.

Yechish.

$$\int\sqrt{2-2x+x^2}dx=\int\sqrt{x^2-2x+1+1}dx=\int\sqrt{(x-1)^2+1}dx\left|\begin{matrix}x-1=t\\ dx=dt\end{matrix}\right|=$$

$$=\int\sqrt{t^2+1}dt==\int\frac{t^2+1}{\sqrt{t^2+1}}dt=\int\frac{t^2dt}{\sqrt{t^2+1}}+\int\frac{dt}{\sqrt{t^2+1}}.$$

Bu yerdagi ikkinchi integral jadval integral. Birinchi integralni bo'laklab integrallash formulasidan foydalanib topamiz:

$$\int\frac{t^2dt}{\sqrt{t^2+1}}=\int t\cdot\frac{tdt}{\sqrt{t^2+1}}=\int t(\sqrt{t^2+1})'dt=$$

$$\int t\cdot d\sqrt{t^2+1}=t\cdot\sqrt{t^2+1}-\int\sqrt{t^2+1}dt.$$

Yana dastlabki integralni hosil qildik, ya'ni

$$\int\sqrt{t^2+1}dt=t\cdot\sqrt{t^2+1}-\int\sqrt{t^2+1}dt+\int\frac{dt}{\sqrt{t^2+1}}.$$

Bundan $2\int\sqrt{t^2+1}dt=t\cdot\sqrt{t^2+1}+\int\frac{dt}{\sqrt{t^2+1}}$

yoki

$$\int\sqrt{t^2+1}dt=\frac{t}{2}\sqrt{t^2+1}+\frac{1}{2}\ln\left|t+\sqrt{t^2+1}\right|+C$$

kelib chiqadi.

Shunday qilib

$$\int\sqrt{2-2x+x^2}\,dx=\frac{x-1}{2}\cdot\sqrt{2-2x+x^2}+\frac{1}{2}\ln\left|x-1+\sqrt{2-2x+x^2}\right|+C$$

36.4. $\int\frac{dx}{(\alpha x+\beta)\sqrt{ax^2+bx+c}}$ ko'rinishidagi integrallar

Bu integral $\alpha=0$ ($\beta=1$) bo'lganda

$$\int\frac{dx}{\sqrt{ax^2+bx+c}}$$

ko'rinishga ega bo'ladi. Oxirgi integral ax^2+bx+c kvadrat uchhaddan to'liq kvadrat ajratish yo'li bilan

$$\int\frac{dx}{\sqrt{m^2-x^2}}\ \text{yoki}\ \int\frac{dx}{\sqrt{x^2\pm m^2}}$$

jadval integrallaridan biriga keltiriladi.

$\alpha\neq 0$ bo'lganda berilgan integral $\alpha x+\beta=\frac{1}{t}$ almashtirish yordamida $\alpha=0$ bo'lgan holdagi integralga keltiriladi.

4-misol. $\int\frac{dx}{x\sqrt{x^2+4x-4}}$ integral hisoblansin.

Yechish.

$$\int\frac{dx}{x\sqrt{x^2+4x-4}}\left|x=\frac{1}{t},dx=-\frac{dt}{t^2}\right|=\int\frac{-\frac{dt}{t^2}}{\frac{1}{t}\sqrt{\frac{1}{t^2}+\frac{4}{t}-4}}=-\int\frac{dt}{\sqrt{1+4t-4t^2}}=$$

$$\int\frac{dt}{\sqrt{2-(1-2t)^2}}=\frac{1}{2}\int\frac{(1-2t)'dt}{\sqrt{2-(1-2t)^2}}=\frac{1}{2}\int\frac{d(1-2t)}{\sqrt{2-(1-2t)^2}}=\frac{1}{2}\arcsin\frac{1-2t}{\sqrt{2}}+C=$$

$$\frac{1}{2}\arcsin\frac{1-\frac{2}{x}}{\sqrt{2}}+C=\frac{1}{2}\arcsin\frac{x-2}{\sqrt{2}x}+C.$$

36.5. $\int \frac{P_n(x)dx}{\sqrt{ax^2+bx+c}}$ **ko'rinishdagi integrallar, bunda *$P_n(x)$ n* – darajali ko'phad**

Bu integral $0 \le n < 2$ bo'lganda qiyinchiliksiz topiladi. $n \ge 2$ bo'l-ganda

$$\int \frac{P_n(x)dx}{\sqrt{ax^2+bx+c}} \equiv Q_{n-1}(x)\sqrt{ax^2+bx+c} + M\int \frac{dx}{\sqrt{ax^2+bx+c}} \qquad (36.1)$$

ayniyat o'rinli deb faraz qilamiz, bu yerda $Q_{n-1}(x)$- koeffitsientlari noaniq bo'lgan $n-1$-darajali ko'phad, M-topilishi kerak bo'lgan son.

Noaniq koeffitsientlar va M son (36.1) ayniyatni differensiallash hamda tenglik ning o'ng va chap tomonidagi bir xil darajali x lar oldidagi koeffitsientlarni tenglash orqali hosil bo'lgan sistemani yechib topiladi.

5-misol. Ushbu $\int \frac{x^2dx}{\sqrt{x^2-x+1}}$ integral hisoblansin.

Yechish. (36.1) ayniyatni yozamiz:

$$\int \frac{x^2dx}{\sqrt{x^2-x+1}} \equiv (Ax+B)\cdot\sqrt{x^2-x+1} + M\int \frac{dx}{\sqrt{x^2-x+1}}. \qquad (36.1')$$

Ayniyatni har ikkala tomonini differensiallaymiz:

$$\frac{x^2}{\sqrt{x^2-x+1}} \equiv A\sqrt{x^2-x+1} + (Ax+B)\frac{2x-1}{2\sqrt{x^2-x+1}} + \frac{M}{\sqrt{x^2-x+1}}.$$

Tenglikni har ikkala tomonini $2\sqrt{x^2-x+1}$ ga ko'paytirib max-rajlardan qutilamiz:

$2x^2 \equiv 2A(x^2-x+1)+(Ax+B)(2x-1)+2M$ yoki

$2x^2 \equiv (2A+2A)x^2+(-2A-A+2B)x+2A-B+2M;$

$2x^2 \equiv 4Ax^2+(-3A+2B)x+2A-B+2M.$

x ning bir xil darajalari oldidagi koeffitsientlarni tenglashtirib topamiz:

$$\left.\begin{array}{l}4A=2,\\ -3A+2B=0,\\ 2A-B+2M=0\end{array}\right\}\quad A=\frac{1}{2},\ 2B=3A=\frac{3}{2},\ B=\frac{3}{4},\ 2M=B-2A=\frac{3}{4}-1=-\frac{1}{4}\quad M=-\frac{1}{8}.$$

A, B, va M ning topilgan qiymatlarini (36.1′) tenglikka qo'ysak

$$\int\frac{x^2dx}{\sqrt{x^2-x+1}}=(\frac{1}{2}x+\frac{3}{4})\sqrt{x^2-x+1}-\frac{1}{8}\int\frac{dx}{\sqrt{x^2-x+1}}$$

bo'ladi. Oxirgi integralni hisoblaymiz:

$$\int\frac{dx}{\sqrt{x^2-x+1}}=\int\frac{d(x-\frac{1}{2})}{\sqrt{(x-\frac{1}{2})^2+\frac{3}{4}}}=\ln\left|x-\frac{1}{2}+\sqrt{(x-\frac{1}{2})^2+\frac{3}{4}}\right|=$$

$$=\ln\left|2x-1+2\sqrt{x^2-x+1}\right|.$$

Shunday qilib

$$\int\frac{x^2dx}{\sqrt{x^2-x+1}}=\frac{2x+3}{4}\sqrt{x^2-x+1}-\frac{1}{8}\ln\left|2x-1+2\sqrt{x^2-x+1}\right|+C$$

tenglikka ega bo'ldik.

6-misol. Ushbu $\int\frac{x^3-x+1}{\sqrt{x^2+2x+2}}dx$ integral topilsin.

Yechish. (36.1) ayniyatni yozamiz.

$$\int\frac{x^3-x+1}{\sqrt{x^2+2x+2}}dx=(Ax^2+Bx+C)\sqrt{x^2+2x+2}+M\int\frac{dx}{\sqrt{x^2+2x+2}}\qquad(36.1'')$$

Tenglikning ikkala tomonini differensiallaymiz:

$$\frac{x^3-x+1}{\sqrt{x^2+2x+2}}=(2Ax+B)\sqrt{x^2+2x+2}+\frac{2x+2}{2\sqrt{x^2+2x+2}}\cdot(Ax^2+Bx+C)+\frac{M}{\sqrt{x^2+2x+2}}.$$

Buni $\sqrt{x^2+2x+2}$ ga ko'paytirib maxrajlardan qutilamiz:

$$x^3-x+1=(2Ax+B)(x^2+2x+2)+(x+1)(Ax^2+Bx+C)+M \text{ yoki}$$

$$x^3-x+1=(2A+A)x^3+(B+4A+B+A)x^2+(4A+2B+C+B)x+2B+C+M$$

yoki

$$x^3-x+1=3Ax^3+(5A+2B)x^2+(4A+3B+C)x+2B+C+M.$$

x ning bir xil darajalri oldidagi koeffitsientlarni tenglashtirib

$$\begin{cases} 3A=1, \\ 5A+2B=0, \\ 4A+3B+C=-1, \\ 2B+C+M=1 \end{cases}$$

sistemani hosil qilamiz. Sistemani yechib

$$A=\frac{1}{3},\ B=-\frac{5A}{2}=-\frac{5}{6},\ C=-1-4A-3B=-1-\frac{4}{3}+\frac{5}{2}=\frac{1}{6},$$

$$M=1-2B-C=1+\frac{5}{3}-\frac{1}{6}=\frac{5}{2}$$

ga ega bo'lamiz.

Koeffitsientlarning topilgan qiymatlarini (36.1′′) ga qo'yib quyidagiga ega bo'lamiz:

$$\int\frac{x^3-x+1}{\sqrt{x^2+2x+2}}dx=(\frac{1}{3}x^2-\frac{5}{6}x+\frac{1}{6})\sqrt{x^2+2x+2}+\frac{5}{2}\int\frac{d(x+1)}{\sqrt{(x+1)^2+1}}=$$

$$=\frac{2x^2-5x+1}{6}\sqrt{x^2+2x+2}+\frac{5}{2}\ln\left|x+1+\sqrt{x^2+2x+2}\right|+C.$$

36.6. Eyler almashtirishlari

$\int R\left(x,\sqrt{ax^2+bx+c}\right)dx,(a\neq 0)$ ko'rinishdagi integralni ratsionallashtirish uchun Eyler qo'yidagi uchta almashtirishni taklif etgan.

a) a>0 bo'lsa

$$\sqrt{ax^2+bx+c}=\pm\sqrt{a}x+t$$

almashtirishni olinadi. Ildiz oldida plyus ishorasini olib

$$\sqrt{ax^2+bx+c}=\sqrt{a}x+t \qquad (36.2)$$

tenglikka ega bo'lamiz. Tenglikni ikkala tomonini kvadratga ko'tarsak

$ax^2+bx+c=ax^2+2\sqrt{a}xt+t^2,\ bx-2\sqrt{a}tx=t^2-c,\ (b-2\sqrt{a}t)x=t^2-c$

va bundan $x=\dfrac{t^2-c}{b-2\sqrt{a}t}$ kelib chiqadi. x ning ushbu qiymatini (36.2) tenglikka qo'yib

$\sqrt{ax^2+bx+c}=\sqrt{a}\dfrac{t^2-c}{b-2\sqrt{a}t}+t$ ni topamiz. x, $\sqrt{ax^2+bx+c}$ va dx larni t orqali ifodalangan ratsional qiymatlarini berilgan integralga qo'yib uni ratsionallashtiramiz.

7-misol. $\int\dfrac{dx}{x\sqrt{x^2+4}}$ integral hisoblansin.

Yechish. $a=1>0$ bo'lgani uchun $\sqrt{x^2+4}=-x+t(\sqrt{x^2+4}+x=t)$ almashtirish olamiz.

U holda $x^2+4=(-x+t)^2=x^2-2xt+t^2,\ 2xt=t^2-4,$

$$x=\frac{t^2-4}{2t},dx=\left(\frac{t^2-4}{2t}\right)'dt=\frac{1}{2}\cdot\frac{2t\cdot t-(t^2-4)\cdot 1}{t^2}dt=$$

$$=\frac{(t^2+4)dt}{2t^2},\sqrt{x^2+4}=-\frac{t^2-4}{2t}+t=\frac{t^2+4}{2t}$$

bo'lishini hisobga olsak

$$\int\frac{dx}{x\sqrt{x^2+4}}=\int\frac{\frac{(t^2+4)dt}{2t^2}}{\frac{t^2-4}{2t}\cdot\frac{t^2+4}{2t}}=2\int\frac{dt}{t^2-4}=2\cdot\frac{1}{2\cdot 2}\ln\left|\frac{t-2}{t+2}\right|+C=$$

$$=\frac{1}{2}\ln\left|\frac{\sqrt{x^2+4}+x-2}{\sqrt{x^2+4}+x+2}\right|+C.$$

b) $c>0$ bo'lsin. Bu holda Eyler

$$\sqrt{ax^2+bx+c}=xt\pm\sqrt{c}$$

almashtirishni taklif etgan. Oxirgi tenglikni har ikkala tomonini kvadratga ko'tarsak

$$ax^2+bx+c=x^2t^2\pm 2xt\sqrt{c}+c,\ ax^2-t^2x^2=$$
$$=\pm 2t\sqrt{c}\,x-bx,\ (a-t^2)x=\pm 2\sqrt{c}\,t-b$$

bo'ladi. $\sqrt{c}$ oldida plyus ishorani olib x ni topamiz,

$$x=\frac{2\sqrt{c}t-b}{a-t^2}.$$

dx va $\sqrt{ax^2+bx+c}$ larni t orqali ifodalab berilgan integralga , x, dx va $\sqrt{ax^2+bx+c}$ ning t orqali qiymatlarini qo'ysak integral ratsionallashadi.

d) x_1 va x_2 ax^2+bx+c kvadrat uchhadning haqiqiy ildizlari bo'lganda

$$\sqrt{ax^2+bx+c}=(x-x_1)t$$

almashtirishni olamiz. U holda $ax^2+bx+c=a(x-x_1)(x-x_2)$ bo'lgani uchun

$$\sqrt{a(x-x_1))(x-x_2)}=(x-x_1)t\ ,$$
$$a(x-x_1)(x-x_2)=(x-x_1)^2t^2,$$
$$a(x-x_2)=(x-x_1)t^2,$$
$$ax-t^2x=ax_2-x_1t^2$$

va bundan $x=\dfrac{ax_2-x_1t^2}{a-t^2}$ kelib chiqadi.

Natijada

$$dx=\left(\frac{ax_2-x_1t^2}{a-t^2}\right)'dt=\frac{-2x_1t(a-t^2)+(ax_2-x_1t^2)2t}{(a-t^2)^2}dt=\frac{2a(x_2-x_1)t}{(a-t^2)^2}dt,$$

$$\sqrt{ax^2+bx+c}=\left(\frac{ax_2-x_1t^2}{a-t^2}-x_1\right)t=\frac{a(x_2-x_1)}{a-t^2}\cdot t$$

tenglikka ega bo'lamiz. x, dx va $\sqrt{ax^2+bx+c}$ larning t orqali qiymatlarini berilgan integralga qo'ysak ratsional funksiyaning integrali hosil bo'ladi.

8-misol. $\int\dfrac{dx}{\sqrt{x^2-2x-8}}$ integral hisoblansin.

Yechish. x^2-2x-8 kvadrat uchhad $x_1=-2$, $x_2=4$ haqiqiy ildizlarga ega.

$$\sqrt{x^2-2x-8}=(x+2)\,t$$

almashtirish olamiz. U holda $x^2-2x-8=(x+2)(x-4)$ bo'lgani uchun

$$\sqrt{(x+2)(x-4)}=(x+2)\,t,\ (x+2)(x-4)=(x+2)^2t^2,$$

$$x-4=(x+2)t^2=xt^2+2t^2,\ (1-t^2)x=2t^2+4,$$

$$x=\frac{2t^2+4}{1-t^2},\quad dx=\frac{4t(1-t^2)-(2t^2+4)(-2t)}{(1-t^2)^2}dt=\frac{12tdt}{(1-t^2)^2},$$

$$\sqrt{x^2-2x-8}=\left(\frac{2t^2+4}{1-t^2}+2\right)\cdot t=\frac{6t}{1-t^2}$$

ga ega bo'lamiz. Bularni berilgan integralga qo'yib uni topamiz.:

$$\int\frac{dx}{\sqrt{x^2-2x-8}}=\int\frac{\frac{12tdt}{(1-t^2)^2}}{\frac{6t}{1-t^2}}=2\int\frac{dt}{1-t^2}=2\cdot\frac{1}{2\cdot1}\ln\left|\frac{1+t}{1-t}\right|+C=\ln\left|\frac{1+t}{1-t}\right|+C.$$

$$t=\frac{\sqrt{x^2-2x-8}}{x+2}=\frac{\sqrt{(x+2)(x-4)}}{x+2}=\sqrt{\frac{(x+2)(x-4)}{(x+2)^2}}=\sqrt{\frac{x-4}{x+2}}$$

ni o‘z o‘rniga qo‘yib va tegishli algebraik almashtirishlarni bajarib

$$\int \frac{dx}{\sqrt{x^2-2x-8}} = \ln\left|\frac{1+\sqrt{\frac{x-4}{x+2}}}{1-\sqrt{\frac{x-4}{x+2}}}\right| + C = \ln\left|\frac{\sqrt{x+2}+\sqrt{x-4}}{\sqrt{x+2}-\sqrt{x-4}}\right| + C =$$

$$= \ln\left|\frac{(\sqrt{x+2}+\sqrt{x-4})^2}{(\sqrt{x+2})^2-(\sqrt{x-4})^2}\right| + C = \ln\left|\frac{x+2+2\sqrt{(x+2)(x-4)}+x-4}{x+2-x+4}\right| + C =$$

$$= \ln\left|\frac{2x-2+2\sqrt{x^2-2x-8}}{6}\right| + C = \ln\left|x-1+\sqrt{x^2-2x-8}\right| + C_1; (C_1 = C - \ln_3)$$

ga ega bo‘lamiz.

36.7. Binomial differensialni integrallash

$x^m(a+bx^n)^p dx$ ifoda binomial differensial deyiladi, bunda a, b o‘zgarmas sonlar, m, n, p-ratsional sonlar.

Binomial differensialning integrali

$$\int x^m(a+bx^n)^p dx \qquad (36.3)$$

ni topish talab etilsin. Bu integral quyidagi hollardagina ratsional funksiyaning integraliga keladi.

1) p-butun son. Bu holda m va n ratsional sonlar maxrajlarining eng kichik umumiy karrasini k orqali belgilab (36.3) integralda

$$x=t^k$$

almashtirishni bajarilsa, integral ratsional funksiyaning integraliga keladi.

9-misol. $\int x^{\frac{1}{3}}(1-2x^{\frac{1}{2}})^3 dx$ integral hisoblansin.

Yechish. $m=\frac{1}{3}, n=\frac{1}{2}, p=3$ butun son. 3 va 2 ning eng kichik umumiy karralisi 6 bo‘lganligi sababli $x=t^6$, $dx=6t^5\,dt$ almashtirish olamiz. U holda

$$\int x^{\frac{1}{3}}(1-2x^{\frac{1}{2}})^3 dx = \int (t^6)^{\frac{1}{3}}\left[1-2(t^6)^{\frac{1}{2}}\right]^3 6t^5dt = 6\int t^2(1-2t^3)^3t^5dt =$$

$$=6\int t^7(1-3\cdot 2t^3+3\cdot 4t^6-8t^9)dt = 6\int (t^7-6t^{10}+12t^{13}-8t^{16})dt =$$

$$6\left(\frac{t^8}{8}-6\cdot\frac{t^{11}}{11}+12\frac{t^{14}}{14}-8\frac{t^{17}}{17}\right)+C=\frac{3}{4}t^8-\frac{36}{11}t^{11}+\frac{36}{7}t^{14}+\frac{48}{17}t^{17}+C=$$

$$=\frac{3}{4}x^{\frac{4}{3}}-\frac{36}{11}x^{\frac{11}{6}}+\frac{36}{7}x^{\frac{7}{3}}-\frac{48}{17}x^{\frac{17}{6}}+C=\frac{3}{4}x\sqrt[3]{x}-\frac{36}{11}x\sqrt[6]{x^5}-\frac{36}{7}x^2\sqrt[3]{x}-\frac{48}{17}x^2\sqrt[6]{x^5}+C.$$

2) $\frac{m+1}{n}$ butun son bo‘lsin. Bu holda (36.3.) integralda $x=t^{\frac{1}{n}}$, $dx=\frac{1}{n}t^{\frac{1}{n}-1}dt$ almashtirish bajarib uni

$$\int x^m(a+bx^n)^p dx = \int t^{\frac{m}{n}}(a+bt)^p\frac{1}{n}t^{\frac{1}{n}-1}dt=\frac{1}{n}\int (a+bt)^p t^{\frac{m+1}{n}-1}dt$$

ko‘rinishga keltiramiz. p ratsional sonning maxraji s bo‘lganda eng so‘nggi integralda

$$(a+bt)^{\frac{1}{s}}=z$$

almashtirishni olsak u ratsional funksiyaning integraliga keladi. Demak, bu holda berilgan integral $a+bx^n=z^s$ almashtirish yordamida ratsionallashtiriladi.

10- misol. $\int x^5\sqrt[3]{(1+x^3)^2}dx$ integral topilsin.

Yechish. Bu integralni $\int x^5(1+x^3)^{\frac{2}{3}}dx$ ko‘rinishda yozsak $m=5$, $n=3$, $p=\frac{2}{3}$, $s=3$ bo‘lib $\frac{m+1}{n}=\frac{5+1}{3}=2$ butun sondir. Shuning uchun $(1+x^3)^{\frac{1}{3}}=z$ yoki $1+x^3=z^3$ almashtirish olamiz. U holda

$x^3=z^3-1$, $x=(z^3-1)^{\frac{1}{3}}$, $dx=\frac{1}{3}(z^3-1)^{\frac{1}{3}-1}\cdot 3z^2dz=z^2(z^3-1)^{-\frac{2}{3}}dz$

bo'ladi. Demak,

$$\int x^5(1+x^3)^{\frac{2}{3}}dx=\int (z^3-1)^{\frac{5}{3}}\cdot z^2\cdot z^2(z^3-1)^{-\frac{2}{3}}dz=\int (z^3-1)z^4dz=\int (z^7-z^4)dz=$$

$$=\frac{1}{8}z^8-\frac{1}{5}z^5+C=z^5\left(\frac{z^3}{8}-\frac{1}{5}\right)+C=(1+x^3)^{\frac{5}{3}}\left(\frac{1+x^3}{8}-\frac{1}{5}\right)+C.$$

3) $\frac{m+1}{n}+p$ butun son bo'lsin. (36.3) integral $x=t^{\frac{1}{n}}$ almashtirish bilan ushbu $\frac{1}{n}\int (a+bt)^p t^{\frac{m+1}{n}-1}dt=\frac{1}{n}\int (a\cdot t^{-n}+b)^p t^{\frac{m+1}{n}+p-1}dt$ ko'rinishga keladi.

Agar keyingi integralda $(at^{-n}+b)^{\frac{1}{s}}=z$ (yoki $at^{-n}+b=z^s$) almashtirish olinsa u ratsional funksiyaning integraliga keladi, bunda *s*, *p* ratsional sonning maxraji.

11- misol. $\int \frac{dx}{x^2(2+x^3)^{\frac{5}{3}}}$ integral hisoblansin.

Yechish. Integralni $\int x^{-2}(2+x^3)^{-\frac{5}{3}}dx$ ko'rinishda yozsak $m=-2$, $n=3$, $p=-\frac{5}{3}$, $s=3$ bo'lib,

$\frac{m+1}{n}+p=\frac{-2+1}{3}-\frac{5}{3}=-2$ butun son. Shuning uchun $2x^{-3}+1=z^3$ $\left((2x^{-3}+1)^{\frac{1}{3}}=z\right)$ almashtirish olamiz. U holda

$$2x^{-3}=z^3-1,\quad x^{-3}=\frac{z^3-1}{2},\quad x=\left(\frac{z^3-1}{2}\right)^{-\frac{1}{3}},$$

$$dx = -\frac{1}{3}\left(\frac{z^3-1}{2}\right)^{-\frac{1}{3}-1}\cdot\frac{3}{2}z^2dz = -\frac{1}{2}\left(\frac{z^3-1}{2}\right)^{-\frac{4}{3}}\cdot z^2\cdot dz$$

bo'lganligidan

$$\int\frac{dx}{x^2(2+x^3)^{\frac{5}{2}}} = \int x^{-2}(2+x^3)^{-\frac{5}{3}}dx = -\int\left(\frac{z^3-1}{2}\right)^{\frac{2}{3}}\left(2+\frac{2}{z^3-1}\right)^{-\frac{5}{3}}\cdot\frac{1}{2}\left(\frac{z^3-1}{2}\right)^{-\frac{4}{3}}\cdot z^2dz =$$

$$= -\frac{1}{2}\int z^2\cdot\frac{(z^3-1)^{\frac{2}{3}}}{2^{\frac{2}{3}}}\cdot\left(\frac{2z^3}{z^3-1}\right)^{-\frac{5}{3}}\cdot\frac{(z^3-1)^{-\frac{4}{3}}}{2^{-\frac{4}{3}}}dz = -\frac{1}{2}\int z^2\cdot\frac{(z^3-1)^{\frac{2}{3}}}{2^{\frac{2}{3}}}\cdot\frac{(z^3-1)^{\frac{5}{3}}}{2^{\frac{5}{3}}\cdot z^5}\cdot\frac{(z^3-1)^{-\frac{4}{3}}}{2^{-\frac{4}{3}}}dz =$$

$$= -\frac{1}{2}\int\frac{(z^3-1)^{\frac{2}{3}+\frac{5}{3}-\frac{4}{3}}}{z^3 2^{\frac{2}{3}+\frac{5}{3}-\frac{4}{3}}}dz = -\frac{1}{2}\int\frac{(z^3-1)}{2\cdot z^3}dz = -\frac{1}{4}\int\frac{z^3-1}{z^3}dz = -\frac{1}{4}\int(1-z^{-3})dz = -\frac{1}{4}\left(z-\frac{z^{-3+1}}{-3+1}\right)+C =$$

$$= -\frac{1}{4}\left(z-\frac{z^{-2}}{-2}\right)+C = -\frac{1}{4}\left(z+\frac{1}{2z^2}\right)+C = -\frac{1}{4}\left[(2x^{-3}+1)^{\frac{1}{3}}+\frac{1}{2(2x^{-3}+1)^{\frac{2}{3}}}\right]+C =$$

$$= -\frac{1}{4}\left[\left(\frac{2}{x^3}+1\right)^{\frac{1}{3}}+\frac{1}{2\left(\frac{2}{x^3}+1\right)^{\frac{2}{3}}}\right]+C = -\frac{1}{4}\left[\frac{(2+x^3)^{\frac{1}{3}}}{x}+\frac{x^2}{2(2+x^3)^{\frac{2}{3}}}\right]+C =$$

$$= -\frac{1}{4}\left[\frac{\sqrt[3]{2+x^3}}{x}+\frac{x^2}{2\sqrt[3]{(2+x^3)^2}}\right]+C$$

bo'ladi.

36.8. $\int R(x,\sqrt{a^2-x^2})dx, \int R(x,\sqrt{a^2+x^2})dx, \int R(x,\sqrt{x^2-a^2})dx,$ ko'rinishdagi integrallar

$\int R(x,\sqrt{a^2-x^2})dx$ ko'rinishdagi integrallar $x=a\sin t$ almashtirish yordamida,

$\int R(x,\sqrt{a^2+x^2})dx$ ko'rinishdagi integrallar $x=atgt$ almashtirish yordamida ,

$\int R(x,\sqrt{x^2-a^2})dx$ ko'rinishdagi integrallar $x=\frac{a}{\cos t}$ almashtirish yordamida ratsional funksiyaning integrallariga keltiriladi.

12- misol. $\int \frac{\sqrt{4-x^2}}{x^2}dx$ integral hisoblansin.

Yechish.

$$\int \frac{\sqrt{4-x^2}}{x^2}dx \left| \begin{matrix} x=2\sin t \\ dx=2\cos t\,dt \end{matrix} \right| = \int \frac{\sqrt{4-4\sin^2 t}}{4\sin^2 t}\cdot 2\cos tdt = \int \frac{2\sqrt{1-\sin^2 t}\cdot 2\cos tdt}{4\sin^2 t} =$$

$$= \int \frac{\sqrt{\cos^2 t}\cos t}{\sin^2 t}dt = \int \frac{\cos^2 t}{\sin^2 t}dt = \int \frac{1-\sin^2 t}{\sin^2 t}dt = \int \left(\frac{1}{\sin^2 t}-1\right)dt = -ctgt - t + C.$$

$\sin t = \frac{x}{2}$ bo'lgani uchun

$$ctgt = \frac{\cos t}{\sin t} = \frac{\sqrt{1-\sin^2 t}}{\sin t} = \frac{\sqrt{1-\left(\frac{x}{2}\right)^2}}{\frac{x}{2}} = \frac{\sqrt{1-\frac{x^2}{4}}}{\frac{x}{2}} = \frac{\sqrt{4-x^2}}{x}, t = \arcsin\frac{x}{2}.$$

Demak, $\int \frac{\sqrt{4-x^2}}{x^2}dx = -\frac{\sqrt{4-x^2}}{x} - \arcsin\frac{x}{2} + C.$

13- misol. $\int \frac{dx}{\sqrt{(4+x^2)^3}}$ integral hisoblansin.

Yechish. $x=2tgt$ almashtirish olamiz. U holda

$$dx = (2tgt)'dt = \frac{2dt}{\cos^2 t}, 4+x^2 = 4+(2tgt)^2 =$$

$$=4+4tg^2 t = 4(1+tg^2 t) = 4\cdot\frac{1}{\cos^2 t}$$

bo'lib,

$$\int\frac{dx}{\sqrt{(4+x^2)^3}}=\int\frac{\dfrac{2dt}{\cos^2 t}}{\sqrt{\left(\dfrac{4}{\cos^2 t}\right)^3}}=\int\frac{\dfrac{2dt}{\cos^2 t}}{\left(\dfrac{2}{\cos t}\right)^3}=\frac{1}{4}\int\cos t dt=\frac{1}{4}\sin t+C=$$

$$\frac{1}{4}\frac{tgt}{\sqrt{1+tg^2 t}}+C=\frac{1}{4}\cdot\frac{\dfrac{x}{2}}{\sqrt{1+\left(\dfrac{x}{2}\right)^2}}+C=\frac{x}{4\sqrt{4+x^2}}+C$$

14- misol. $\int\frac{\sqrt{x^2-4}}{x^3}dx$ integral hisoblansin.

Yechish. $x=\frac{2}{\cos t}$ almashtirish olamiz. U holda

$$dx=\left(\frac{2}{\cos t}\right)'dt=-2\cdot\frac{(\cos t)'}{\cos^2 t}dt=2\frac{\sin t dt}{\cos^2 t},\ \sqrt{x^2-4}=\sqrt{\left(\frac{2}{\cos t}\right)^2-4}=$$

$$\sqrt{\frac{4}{\cos^2 t}-4}=\sqrt{\frac{4-4\cos^2 t}{\cos^2 t}}=\sqrt{\frac{4(1-\cos^2 t)}{\cos^2 t}}=2\sqrt{\frac{\sin^2 t}{\cos^2 t}}=2tgt$$

bo'ladi. Demak,

$$\int\frac{\sqrt{x^2-4}}{x^3}dx=\int\frac{2tgt}{\left(\dfrac{2}{\cos t}\right)^3}\cdot\frac{2\sin t}{\cos^2 t}dt=\frac{1}{2}\int\sin^2 t dt=\frac{1}{2}\int\frac{1-\cos 2t}{2}dt=$$

$$\frac{1}{4}\int(1-\cos 2t)dt=\frac{1}{4}\left(t-\frac{\sin 2t}{2}\right)+C=\frac{1}{4}(t-\sin t\cdot\cos t)+C.$$

$x=\frac{2}{\cos t}$ tenglikdan $\cos t=\frac{2}{x}$, $\quad t=\arccos\frac{2}{x}$,

$\sin t=\sqrt{1-\cos^2 t}=\sqrt{1-\left(\frac{2}{x}\right)^2}=\frac{\sqrt{x^2-4}}{x}$ ga ega bo'lamiz.

Shuning uchun

$$\int\frac{\sqrt{x^2-4}}{x^3}dx=\frac{1}{4}\left(\arccos\frac{2}{x}-\frac{2\sqrt{x^2-4}}{x^2}\right)+C.$$

O'z-o'zini tekshirish uchun savollar

1. Har qanday irratsional funksiyaning integralini ratsionallashtirib bo'ladimi?

2. $\int R(x,x^{\frac{m}{n}},...,x^{\frac{r}{s}})dx$ qanday ratsionallashtiriladi?

3. $\int\sqrt{ax^2+bx+c}\quad dx$ qanday ratsionallashtiriladi?

4. $\int\frac{dx}{(\alpha x+\beta)\sqrt{ax^2+bx+c}}$ qanday ratsionallashtiriladi?

5. $\int\frac{P_n(x)dx}{\sqrt{ax^2+bx+c}}$ qanday ratsionallashtiriladi?

6. Eyler almashtirishlari nimalardan iborat?

7. Binominal differensiallar qanday hollarda ratsional funksiyaga keltiriladi?

8. $\int R(x,\sqrt{a^2-x^2})dx$, $\int R(x,\sqrt{a^2+x^2})dx$, $\int R(x,\sqrt{x^2-a^2})dx$ qanday ratsionallashtiriladi?

Mustaqil yechish uchun mashqlar

1. $\int\frac{\sqrt{x^3}-\sqrt[3]{x}}{6\sqrt[4]{x}}dx.\quad$ *Javob*: $\quad\frac{2}{27}\sqrt[4]{x^9}-\frac{2}{13}\sqrt[12]{x^{13}}+C.$

2. $\int\sqrt{\frac{1-x}{1+x}}\cdot\frac{dx}{x^2}$. *Javob*: $\ln\left|\frac{\sqrt{1-x}+\sqrt{1+x}}{\sqrt{1-x}+\sqrt{1+x}}\right|-\frac{\sqrt{1-x^2}}{x}+C.$

3. $\int\sqrt{2x-x^2}\,dx$. *Javob*: $\frac{1}{2}\left[(x-1)\sqrt{2x-x^2}+\arcsin(x-1)\right]+C.$

4. $\int\frac{dx}{x\sqrt{x^2-x+3}}$. *Javob*: $\frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{x^2-x+3}-\sqrt{3}}{3}+\frac{1}{2\sqrt{3}}\right|+C.$

5. $\int\frac{(x^2+1)dx}{\sqrt{x^2-2x+5}}$. *Javob*: $\frac{1}{2}(x+3)\sqrt{x^2-2x+5}+C.$

6. $\int\frac{dx}{x\sqrt{x^2+4}}$. *Javob*: $\frac{1}{2}\ln\left|\frac{x+\sqrt{x^2+4}-2}{x+\sqrt{x^2+4}-2}\right|+C.$

7. $\int\frac{1-\sqrt{1+x+x^2}}{x\sqrt{1+x+x^2}}dx$. *Javob*: $\ln\left|\frac{2\sqrt{1+x+x^2}-x-2}{x^2}\right|+C.$

8. $\int\frac{dx}{\sqrt{3-2x-x^2}}$. *Javob*: $2\operatorname{arctg}\sqrt{\frac{3+x}{1-x}}+C.$

9. $\int\frac{\sqrt{x}}{\left(1+\sqrt[3]{x}\right)^2}dx$ *Javob*: $\frac{6}{5}\sqrt[6]{x^5}-4\sqrt{x}+18\sqrt[6]{x}-21\operatorname{arctg}\sqrt[6]{x}+\frac{3\sqrt[6]{x}}{\sqrt[3]{x}+1}+C.$

10. $\int\frac{xdx}{\sqrt{1+\sqrt[3]{x^2}}}$. *Javob*: $\frac{3}{7}\sqrt{\left(1+x^{\frac{2}{3}}\right)^7}-\frac{6}{5}\sqrt{\left(1+x^{\frac{2}{3}}\right)^5}+\sqrt{\left(1+x^{\frac{2}{3}}\right)^3}+C$

11. $\int\frac{dx}{x^2\sqrt{2+3x^2}}$. *Javob*: $-\frac{\sqrt{2+3x^2}}{2x}+C.$

12. $\int\frac{\sqrt{a^2-x^2}}{x^2}dx$. *Javob*: $-\frac{\sqrt{a^2-x^2}}{x}-\arcsin\frac{x}{a}+C.$

13. $\int\frac{dx}{x^2\sqrt{1+x^2}}$. *Javob*: $-\frac{\sqrt{1+x^2}}{x}+C.$

14. $\int\frac{\sqrt{x^2-a^2}}{x}dx$. *Javob*: $\sqrt{x^2-a^2}-a\arccos\frac{a}{x}+C.$

37. ANIQ INTEGRAL TUSHUNCHASI

37.1. Aniq integral tushunchasiga keltiruvchi yuza haqidagi masala

Maktab geometriya kursida kesmalar bilan chegaralangan figuralarning yuzlarini, doira hamda uning bo'lagini yuzini topish o'rganiladi. Shuningdek, egri chiziqlar bilan chegaralangan figuraning yuzini topish ham qisman o'rganiladi.

Bu yerda ixtiyoriy yopiq egri chiziq bilan chegaralangan yassi figuraning yuzini topish masalasi bilan jiddiy shug'ullanamiz.

Avvaliga xususiy holni, ya'ni figura Oxy tekisligiga joylashgan bo'lib, yuqoridan uzluksiz $y=f(x)$ $(f(x)\geq 0)$ egri chiziq, quyidan $o\,x$ o'qning $[a,b]$ $(a<b)$ kesmasi va yon tomonlardan $x=a$, $x=b$ vertikal to'g'ri chiziqlar bilan chegaralangan holni qaraymiz. Bu figurani egri chiziqli **trapetsiya** deb ataymiz va $[a,b]$ kesmani uning **asosi** deymiz. Shu egri chiziqli trapetsiyaning yuzini topamiz.

$[a,b]$ kesmani

$$a=x_0<x_1<x_2<\ldots<x_{n-1}<x_n=b$$

nuqtalar bilan n ta ixtiyoriy

$$[x_0,x_1],\ [x_1,x_2],\ldots\ [x_{n-1},x_n]$$

kesmalarga ajratamiz. Bu nuqtalar orqali $o\,y$ ga parallel to'g'ri chiziqlar o'tkazib egri chiziqli trapetsiyani n ta kichik trapetsiyachalarga ajratamiz. U holda qaralayotgan egri chiziqli trapetsiyaning yuzi n ta kichik egri chiziqli trapetsiyachalarning yuzlari yig'indisiga teng bo'lishi ravshan.

Shuning uchun S orqali egri chiziqli trapetsiyaning yuzini ΔS_k orqali asosi $[x_{k-1},\ x_k]$ bo'lgan kichik egri chiziqli trapetsiyaning yuzini belgilasak $S=\Delta S_1+\Delta S_2+\ldots+\Delta S_n$ bo'ladi.

Qisqacha buni $S=\sum\limits_{k=1}^{n}\Delta S_k$ ko'rinishda yozish qabul qilingan.

$$[x_0,x_1],\ [x_1,x_2],\ldots\ [x_{n-1},x_n]$$

kesmalarning har birida bittadan ixtiyoriy nuqta olib ularni $z_1, z_2,...,z_n$ lar orqali belgilaymiz. Bu abssissalarda egri chiziq nuqtalarining ordinatalari $f(z_1), f(z_2),... ,f(z_n)$ larni yasaymiz.

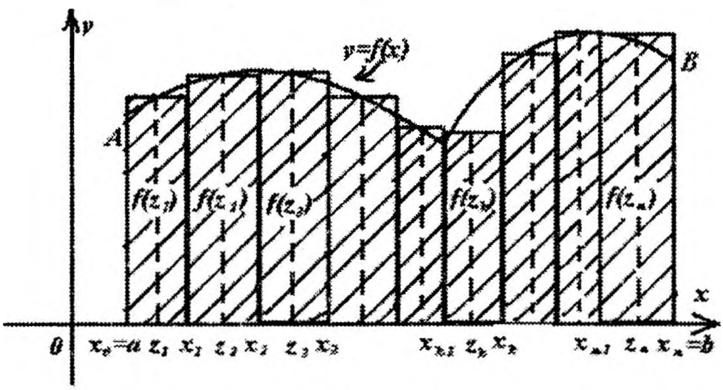
Keyin har bir asosi $[x_{k-1}, x_k]$ $(k=1,2, ... n)$ bo'lgan egri chiziqli trapetsiyachalar yuzini, asosi xuddi shunday, balandligi $f(z_k)$ bo'lgan to'g'ri to'rtburchakning yuzi bilan almashtiramiz. Bu to'g'ri to'rtburchakning yuzi

$$f(z_k)\,\Delta x_k$$

bo'lishi ravshan, bunda $\Delta x_k = x_k - x_{k-1}$ $[x_{k-1}, x_k]$ kesmaning uzunligi (Δx_k asos, $f(z_k)$ -balandlik).

Bu yuzni mos egri chiziqli trapetsiyaning yuzini taqribiy qiymati deb qabul qilsak $\Delta S_k \approx f(z_k)\,\Delta x_k$ va $S \approx f(z_1)\,\Delta x_1 + f(z_2)\,\Delta x_2 + ... + f(z_n)\cdot\Delta x_n$ yoki qisqacha

$$S = \sum_{k=1}^{n} f(z_k)\Delta x_k \quad (37.1) \text{ bo'ladi.}$$



147-chizma.

Agar λ orqali $\Delta x_1, \Delta x_2, ..., \Delta x_n$ larning eng kattasini belgilasak λ kichrayganda (37.1) formulaning aniqlik darajasi ortadi. Shuning uchun (37.1) ning o'ng tomonidagi ifodaning $\lambda \to 0$ dagi limitini S ning **aniq qiymati** deb qabul qilish mumkin, ya'ni

$$S = \lim_{\lambda \to 0} \sum_{k=1}^{n} f(z_k)\Delta x_k . \qquad (37.2)$$

Shunday qilib, egri chiziqli trapetsiyaning yuzini topish masalasi (37.2) ko'rinishdagi yig'indining limitini topishga olib keldi.

Ixtiyoriy yopiq egri chiziq bilan chegaralangan yassi figuraning yuzini topish masalasi ham egri chiziqli trapetsiyaning yuzini topish masalasiga keltirilishini ta'kidlab o'tamiz.

Yuzdan tashqari ko'pgina masalalarning yechimi ham (37.2) ko'rinishdagi limitni topishga kelishini ta'kidlab o'tamiz. Shuning uchun $\sum_{k=1}^{n} f(z_k)\Delta x_k$ ko'rinishdagi yig'indi mazmunan nimani anglatishidan qat'iy nazar uning limintini o'rganamiz.

37.2. Aniq integralning ta'rifi. Integrallanuvchi funksiyalar sinfi

Aniq integral oliy matematikaning eng muhim tushunchalaridan biridir. Yuzlarni, yoylarning uzunliklarini, hajmlarni, ishni, inersiya va statik momentlarni, og'irlik markazi koordinatalarini, yo'lni, bosimni va hokazolarni uning yordamida hisoblash mumkin.

$[a,b]$ kesmada aniqlangan $y=f(x)$ funksiya berilgan bo'lsin. Quyidagi amallarni bajaramiz:

1) $[a,b]$ kesmani

$$a=x_0<x_1<x_2<\ldots<x_{k-1}<x_k<\ldots<x_{n-1}<x_n=b$$

bo'luvchi nuqtalar yordamida n ta «kichik»

$$[x_0,x_1],\ [x_1,x_2],\ldots,\ [x_{k-1},x_k],\ldots,\ [x_{n-1},x_n]$$

kesmalarga ajratib ularning uzunliklarini mos ravishda Δx_1, Δx_2, ..., Δx_n lar orqali belgialymiz.

2) Har bir $[x_{k-1},x_k]$ $(k=1,2,\ldots,n)$ kesmada bittadan ixtiyoriy nuqta tanlab olib ularni mos ravishda $z_1, z_2,\ldots,z_n$ lar orqali belgilaymiz.

3) Tanlangan nuqtalarda funksiyaning

$$f(z_1),\ f(z_2),\ldots,\ f(z_n)$$

qiymatlarini hisoblaymiz.

4) Funksiyaning hisoblangan qiymatini tegishli $[x_{k-1}, x_k]$ kesmachaning uzunligi Δx_k ga ko‘paytirib $f(z_1)\ \Delta x_1$, $f(z_2)\ \Delta x_2, \ldots$, $f(z_k)\Delta x_k, \ldots, f(z_n)\Delta x_n$ ko‘paytmalarni tuzamiz.

5) Tuzilgan ko‘paytmalarni qo‘shamiz va yig‘indini σ_n bilan belgilaymiz:

$$\sigma_n = f(z_1)\ \Delta x_1 + f(z_2)\ \Delta x_2 + \ldots + f(z_k)\Delta x_k + \ldots + f(z_n)\Delta x_n$$

yoki qisqacha $\sigma_n = \sum_{k=1}^{n} f(z_k)\Delta x_k$.

σ_n yig‘indi $f(x)$ funksiya uchun $[a,b]$ kesmada tuzilgan **integral yig‘indi** deb ataladi.

6) $[x_{k-1}, x_k]$ kesmalarning uzunliklaridan eng kattasini λ orqali belgilab uni **bo‘linish odimi** deb ataymiz.

Endi bo‘linishlar soni n ni orttira boramiz $(n \to \infty)$ va bunda $\lambda \to 0$ di deb faraz qilamiz.

1-ta‘rif. Agar $\lambda \to 0$ da σ_n integral yig‘indi $[a,b]$ kesmani qismiy $[x_{k-1}, x_k]$ kesmalarga ajratish usuliga va ularning har biridan z_k nuqtani tanlash usuliga bog‘liq bo‘lmaydigan chekli songa intilsa, u holda shu son $[a,b]$ kesmada $f(x)$ funksiyadan olingan **aniq integral** deyiladi va

$$\int_a^b f(x)dx$$

kabi belgilanadi. Bu integral odatda Riman integrali deb yuritiladi.

Bu yerda $f(x)$-integral ostidagi funksiya, $[a,b]$-kesma integrallash oralig‘i, a va b sonlar integrallashning quyi va yuqori chegarasi, x-integrallash o‘zgaruvchisi deyiladi.

Shunday qilib, aniq integralning ta‘rifidan

$$\int_a^b f(x)dx = \lim_{\lambda \to 0} \sum_{k=1}^{n} f(z_k)\Delta x_k$$

kelib chiqadi.

1-misol. Aniq integralning ta’rifidan foydalanib $\int_0^1 xdx$ integral hisoblansin.

Yechish. Aniq integralaning ta'rifiga binoan

$$\int_0^1 x dx = \lim_{\lambda\to 0}\sigma_n = \lim_{\lambda\to 0}\sum_{k=1}^{n} z_k \Delta x_k .$$

bunda

$\lambda = \max\Delta x_k$, $\quad 0 = x_0 < x_1 < x_2 < ... < x_n = 1,\ x_{k-1} \le z_k \le x_k,\ \Delta x_k = x_k - x_{k-1}.$

$[0,1]$ kesmani $0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n-1}{n}, 1$ nuqtalar yordamida n ta uzunligi $\frac{1}{n}$ ga teng bo'lgan $[0,\frac{1}{n}]$, $[\frac{1}{n},\frac{2}{n}]$, ..., $[\frac{n-1}{n},1]$ "kichik" kesmalarga ajratamiz. Bu kesmalarning uzunligi $n \to \infty$ da 0 intilishi ravshan.

$z_1, z_2, ..., z_n$ nuqtalar sifatida "kichik" kesmalarning o'ng chegaralari $\frac{1}{n}, \frac{2}{n}, ..., 1$ nuqtalarni tanlaymiz.

$z_k = \frac{k}{n}, \Delta x_k = \frac{1}{n}$ ekanini hisobga olib integral yig'indini tuzamiz:

$$\sigma_n = \sum_{k=1}^{n} \frac{k}{n}\cdot\frac{1}{n} = \frac{1}{n^2}\sum_{k=1}^{n} k = \frac{1}{n^2}(1+2+3+...+n) =$$

$$= \frac{1}{n^2}\cdot\frac{n(n+1)}{2} = \frac{1}{2}\cdot\frac{n+1}{n} = \frac{1}{2}(1+\frac{1}{n})$$

Bu integral yig'indisining $n \to \infty$ dagi limiti

$$\lim_{n\to\infty}\sigma_n = \frac{1}{2}\lim_{n\to\infty}(1+\frac{1}{n}) = \frac{1}{2}$$

bo'ladi. Demak

$$\int_0^1 x dx = \frac{1}{2}.$$

2-misol. Ushbu $\int_1^2 \frac{dx}{x}$ integral aniq integralning ta'rifidan foydalanib topilsin.

Yechish. [1,2] kesmani $x_0=1,\ x_1=2^{\frac{1}{n}},\ x_2=2^{\frac{2}{n}},\ \ldots,$ $x_{n-1}=2^{\frac{n-1}{n}},\ x_n=2$ bo'lish nuqtalari yordamida n ta

$$\left[1,2^{\frac{1}{n}}\right],\left[2^{\frac{1}{n}},2^{\frac{2}{n}}\right],\left[2^{\frac{2}{n}},2^{\frac{3}{n}}\right],\ldots,\left[2^{\frac{n-1}{n}},2\right]$$

qismiy kesmalarga ajratamiz. Tabiiyki bu kesmalarning uzunliklari har xil bo'ladi.

Masalan $k-\left[2^{\frac{k-1}{n}},2^{\frac{k}{n}}\right]$ kesmaning uzunligi $\Delta x_k=2^{\frac{k}{n}}-2^{\frac{k-1}{n}}=2^{\frac{k-1}{n}}(2^{\frac{1}{n}}-1)$ bo'lib u $n\to\infty$ da nolga intiladi.

z_k nuqta sifatida $\left[2^{\frac{k-1}{n}},2^{\frac{k}{n}}\right]$ kesmaning o'ng chegarasini olamiz, ya'ni $z_k=x_k=2^{\frac{k}{n}}$.

$$f(x_k)=\frac{1}{z_k}=\frac{1}{2^{\frac{k}{n}}},\quad \Delta x_k=2^{\frac{k-1}{n}}(2^{\frac{1}{n}}-1)$$

ekanini hisobga olib integral yig'indinini tuzamiz.

$$\sigma_n=\sum_{k=1}^{n}f(z_k)\Delta x_k=\sum_{k=1}^{n}\frac{1}{2^{\frac{k}{n}}}\cdot 2^{\frac{k-1}{n}}(2^{\frac{1}{n}}-1)=\sum_{k=1}^{n}\frac{2^{\frac{1}{n}}-1}{2^{\frac{1}{n}}}=\frac{2^{\frac{1}{n}}-1}{2^{\frac{1}{n}}}.$$

Integral yig'indining $n\to\infty$ dagi limitini topamiz:

$$\lim_{n\to\infty}\sigma_n=\lim_{n\to\infty}\frac{1}{2^{\frac{1}{n}}}\cdot\lim_{n\to\infty}\frac{2^{\frac{1}{n}}-1}{\frac{1}{n}}=1\cdot\ln 2=\ln 2.$$

Bu yerda $\lim\limits_{x\to\infty}\frac{a^x-1}{x}=\ln a$ ajoyib limitdan foydalanildi.

Demak, $\int_1^2 \frac{dx}{x} = \lim_{n\to\infty} \sigma_n = \ln 2.$

Bu misollardan ko'rinib turibdiki aniq integrallarni uning ta'rifi bo'yicha hisoblash unchalik oson ish emas ekan.

Aniq integralning ta'rifidan aniq integral mavjud bo'lishi uchun $f(x)$ funksiyaning $[a,b]$ kesmada chegeralangan bo'lishi zarur.

2-ta'rif. Agar chekli $\lim_{\lambda\to 0}\sum_{k=1}^{n} f(z_k)\Delta x_k$ mavjud bo'lsa $f(x)$ funksiya $[a,b]$ kesmada **integrallashuvchi** deyiladi.

37.1-teorema (funksiya integrallashuvchi bo'lishining zaruriy sharti). Agar $f(x)$ funksiya $[a,b]$ kesmada integrallashuvchi bo'lsa, u shu kesmada chegaralangandir.

Isboti. Teskarisini faraz qilamiz, ya'ni $[a,b]$ da integrallashuvchi $f(x)$ funksiya shu kesmada chegaralanmagan bo'lsin. U holda σ_n integral yig'indini z_1, z_2, $z_3,..,z_n$ nuqtalarni tanlash hisobiga istalgancha katta qilish mumkinligini, ya'ni $\lim_{\lambda\to 0}\sigma_n$ limit mavjud bo'lmasligini ko'rsatamiz.

Bu holda $[a,b]$ kesmani istalgan $[x_0,x_1]$, $[x_1,x_2]$, $[x_2,x_3]$, ..., $[x_{n-1},x_n]$ kesmalarga ajratilishini qaramaylik $f(x)$ funksiya $[a,b]$ kesmada chegaralanmaganligi sababli u shu kesmalarning kamida bittasi, masalan $[x_0,x_1]$da chegaralanmagan bo'ladi. Qolgan $[x_1,x_2]$, $[x_2,x_3]$, ..., $[x_{n-1},x_n]$ kesmalarda bittadan ixtiyoriy nuqtalarni olib ularni mos ravishda $z_2, z_3,..,z_n$ lar orqali belgilaymiz va

$$f(z_2)\,\Delta x_2 + f(z_3)\Delta x_3 + \ldots + f(z_n)\Delta x_n = \sigma_n'$$

deb olamiz. $f(x)$ funksiya $[x_0,x_1]$ kesmada chegarlanmaganligi uchun istalgan yetarlicha katta $M>0$ sonni olmaylik $[x_0,x_1]$ kesmada shunday z_1 nuqta mavjud bo'lib,

$$|f(z_1)| \ge \frac{|\sigma_n'| + M}{\Delta x_1} \quad \text{yoki} \quad |f(z_1)|\Delta x_1 \ge |\sigma_n'| + M$$

bo'ladi. Ikkinchi tomondan

$$\sigma_n = f(z_1)\,\Delta x_1 + f(z_2)\,\Delta x_2 + \ldots + f(z_k)\Delta x_k + \ldots + f(z_n)\Delta x_n = f(z_1)\,\Delta x_1 + \sigma_n'$$

bo'lgani uchun $|\sigma_n| = |f(z_1)\,\Delta x_1 + \sigma_n'| \geq |f(z_1)|\,\Delta x_1 - |\sigma_n'| \geq$ $\geq |\sigma_n'| + M - |\sigma_n'| = M$ kelib chiqadi. $|\sigma_n| \geq M$ tengsizlik $\lim\limits_{\lambda \to 0} \sigma_n$ ning mavjud emasligini, ya'ni $f(x)$ funksiyaning $[a,b]$ kesmada integrallanuvchi emasligini ko'rsatadi. Bu teoremaning shartiga zid. Bu ziddiyatga $[a,b]$ kesmada integrallanuvchi $f(x)$ funksiya shu kesmada chegaralanmagan deb qilgan noto'g'ri farazimiz oqibatida keldik.

Keltirilgan teorema faqatgina zaruriy shartdan iborat bo'lib, u yetarli emas, ya'ni funksiyaning chegaralanganligidan uning integrallanuvchiligi kelib chiqmaydi.

Masalan, Dirixle funksiyasi

$$f(x) = \begin{cases} 1, & \text{agar } x \text{ ratsional bo'lsa,} \\ 0, & \text{agar } x \text{ irratsional bo'lsa} \end{cases}$$

ni $[0,1]$ kesmada qarasak u shu kesmada chegarlangan$(|f(x)| \leq 1)$. Ammo bu funksiya $[0,1]$ kesmada integrallanuvchi emas.

Haqiqatan, agar $[0,1]$ kesma kichik kismalarga ajratilganda $z_1, z_2, z_3, \ldots, z_n$ nuqtalar sifatida ratsional sonlar olinsa integral yig'indi

$$\sigma_n = \sum_{k=1}^{n} f(z_k)\Delta x_k = \sum_{k=1}^{n} 1 \cdot \Delta x_k = \Delta x_1 + \Delta x_2 + \ldots + \Delta x_n = 1$$

bo'ladi va $z_1, z_2, z_3, \ldots, z_n$ lar sifatida irratsional sonlar olinsa integral yig'indi

$$\sigma_n = \sum_{k=1}^{n} f(z_k)\Delta x_k = \sum_{k=1}^{n} 0 \cdot \Delta x_k = 0$$

bo'ladi. Bu tengliklardan σ_n integral yig'indi $\lambda \to 0$ da limitga ega emasligi, ya'ni Dirixle funksiyasi $[0,1]$ kesmada integrallanuvchi emasligi kelib chiqadi.

Bu misol shuni ko'rsatadiki, hatto chegaralangan funksiyalarning ham aniq integrallari mavjud bo'lmasligi mumkin ekan.

Endi qanaqa funksiyalarni aniq integrali (Riman integrali) har doim mavjud bo'ladi degan savolga javob izlaymiz. Buning uchun avvalo to'plamlar nazariyasidan kerakli ba'zi-bir tushunchalarni keltiramiz.

Ma'lumki chekli sondagi elementlarga ega to'plam **chekli** to'plam deyiladi.

Agar istalgan n natural son uchun X to'plamning elementlari soni shu n sondan katta bo'lsa X to'plam **cheksiz** to'plam deyiladi.

Masalan, natural, butun, ratsional va haqiqiy sonlar to'plami cheksiz to'plamlardir.

Elementlari orasida o'zaro bir qiymatli moslik o'rnatish mumkin bo'lgan A va B to'plamlar **ekvivalent** to'plamlar deyiladi va $A \sim B$ kabi belgilanadi. A va B ekvivalent to'plamlar bo'lganda A ning har bir a elementiga biror qonun yoki qoida yordamida B to'plamning aniq b elementi mos keladi. Jumladan A to'plamning har xil a_1, a_2 elementlariga B to'pamning har xil b_1, b_2 elementlari mos keladi.

Natural sonlar toplami N ga ekvivalent to'plam **sanoqli** to'plam deyiladi. Ta'rifga binoan natural sonlar to'plamining o'zi ham sanoqli to'plam ekanligi kelib chiqadi. Shuningdek $N_1 = \{\frac{1}{n}\}$ sonlar to'plami ham sanoqli, chunki N ning har bir n elementiga N_1 to'plamning $\frac{1}{n}$ elementini mos qo'yish mumkin.

Shunday qilib sanoqli to'plamning elementlarini nomerlash mumkin ekan. Shuning uchun sanoqli to'plamni ba'zan $X = \{x_1, x_2, ..., x_n, ...\}$ ko'rinishda yoziladi. Sanoqli bo'lmagan to'plamlarni **sanoqsiz** to'lplam deb ataladi.

Ratsional sonlar to'lplami ham sanoqli to'plam ekanligini hamda sanoqli sondagi sanoqli (chekli) to'lplamlarning yig'indisi ham sanoqli to'plam bo'lishini ta'kidlab o'tamiz.

Haqiqiy sonlar to'plami sanoqsiz to'plamdir, ya'ni haqiqiy sonlar to'plamini elementlarini nomerlab bo'lmaydi.

3-ta'rif. Agar istalgan $\varepsilon > 0$ son uchun A to'plamni barcha elementlarini o'zida saqlovchi va uzunliklarining yig'indisi ε dan oshmaydigan chekli yoki sanoqli intervallar mavjud bo'lsa, u holda A to'plamning **lebeg o'lchovi** nolga teng deb ataladi.

Masalan, chekli yoki sanoqli to'plamlarning lebeg o'lchovi nolga teng.

Haqiqatdan E sanoqli yoki chekli to'plam bo'lsin. U holda bu to'plamning elementlarini nomerlash mumkin. $x_1, x_2, x_3, ..., x_n, ...$ shu to'plamning elementlari va ε istalgan musbat son bo'lsin. Har bir x_n elementni uzunligi $\varepsilon \cdot 2^{-n}$ bo'lgan interval bilan qoplaymiz, ya'ni elementni shunday intervalga joylashtiramiz.

U holda cheksiz kamayuvchi geometrik progressiyaning hadlari yig'indisini topish formulasiga binoan bu intervallarni uzunliklarini yig'indisi

$$\frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} + \frac{\varepsilon}{2^3} + ... = \varepsilon$$

dan oshmaydi.

Shuni aytish joizki lebeg o'lchovi *0* ga teng bo'lgan to'plamlar orasida sanoqsiz to'plamlar ham bor.

Qanaqa funksiyalarning integrallari mavjud degan savolga uzil-kesil javobni quyidagi teorema beradi.

Lebeg teoremasi. Chekli $[a,b]$ kesmada $f(x)$ funksiya intgerallanuvchi bo'lishi uchun uning $[a,b]$ kesmada chegaralangan va shu kesmaning lebeg o'lchovi nolga teng to'plamidan tashqari barcha nuqtalarida uzluksiz bo'lishi zarur va yetarlidir.

Shunday qilib, Lebeg teoremasiga ko'ra $[a,b]$ kesmada chegaralangan va shu kesmaning chekli yoki sanoqli nuqtalarida uzilishga ega bo'lib qolgan barcha nuqtalarida uzluksiz funksiya shu kesmada integrallanuvchi bo'lar ekan.

Bu teorema quyidagi natijalarga ega.

1-natija. $[a,b]$ kesmada uzluksiz funksiya shu kesmda integrallanuvchidir.

2-natija. $[a,b]$ kasmada chegaralangan va shu kesmaning chekli yoki sanoqli sondagi nuqtalarida uzilishga ega bo'lib uning qolgan barcha nuqtalarida uzluksiz funksiya shu kesmada integrallanuvchidir.

Bu natijaga binoan $[a,b]$ kesmaning chekli yoki sanoqli sondagi nuqtalarida teng bo'lmay qolgan barcha nuqtalarida teng bo'lgan ikkita *f(x)* va $\varphi(x)$ funksiyalardan biri shu kesmada integrallanuvchi bo'lsa, u holda ikkinchisi ham integrallanuvchi bo'lib ularning integrallari teng bo'ladi.

3-natija. $[a,b]$ kesmada chegaralangan va monoton funksiya shu kesmada integrallanuvchidir.

Haqiqatdan $[a,b]$ kesmada chegaralangan monoton funksiya shu kesmada chekli yoki sanoqli sondagi uzilish nuqtalariga ega bo'lishi mumkin.

Shuni aytish joizki $[a,b]$ kesmada chegaralangan va undagi sanoqsiz to'plamda uzilishga ega bo'lib uning qolgan barcha nuqtalarida uzluksiz bo'lgan funksiyalar orasida integrallanuvchi bo'lganlari ham bo'lmaganlari ham mavjud.

Shunday qilib *[a,b]* kesmaning cheksiz ko'p nuqtalarida uzilishga ega bo'lgan chegarlangan funksiyalar orasida integrallanuvchi bo'lganlari ham bo'lmaganligi ham mavjud. Integrallanuvchi bo'lmaganiga Dirixle funksiyani misol keltirish mumkin.

Chegarlangan va cheksiz ko'p nuqtalarda uzilishga ega bo'lib integrallanuvchi funksiyaga

$$f(x)=\begin{cases} 1,\text{agar}\,\dfrac{1}{2n}<x\le\dfrac{1}{2n-1}\,\text{bo'lsa},\\ -1,\text{agar}\,\dfrac{1}{2n+1}<x\le\dfrac{1}{2n}\,\text{bo'lsa}, n=1,2,3,...\\ 0,\text{agar}\;\; x=0 \quad \text{bo'lsa}\end{cases}$$

funksiyani misol keltirish mumkin. Bu funksiya *[0,1]* kesmada chegaralangan $(|f(x)|\le 1)$ va barcha $x_n=\dfrac{1}{n}, n=1,2,3,...$ sanoqli nuqtalarda birinchi tur uzilishga ega. Lebeg teoremasiga ko'ra bu funksiya $[0,1]$ kesmada integrallanuvchi.

Endi aniq integralning ta'rifidan bevosita kelib chiqadigan xos-salarini keltiramiz.

1-izoh. Aniq integralning qiymati integrallash o'zgaruvchisining qanday harf bilan belgilanishiga bog'liq emas. Masalan:

$$\int_a^b f(x)dx = \int_a^b f(t)dt = \int_a^b f(z)dz.$$

2-izoh. Aniq integralning chegaralari almashtirilsa, integralning ishorasi o'zgaradi.

$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

3-izoh. Chegaralari teng bo'lgan aniq integralning qiymati *0* ga teng.

$$\int_a^a f(x)dx = 0.$$

Aniq integralning ta'rifidan foydalanib egri chiziqli trapetsiyaning yuzini ifodalovchi (37.2) tenglikni

$$S = \int_a^b f(x)dx$$

ko'rinishda yozish mumkin. Boshqacha aytganda $\int_a^b f(x)dx$ aniq integralning **geometrik ma'nosi** egri chiziqli trapetsiyaning **yuzini** ifodalar ekan.

O'z-o'zini tekshirish uchun savollar

1. Egri chiziqli trapetsiya nima?
2. Aniq integralni ta'riflang.
3. Aniq integralni geometrik ma'nosi nimani ifodalaydi?
4. Funksiya integrallanuvchi bo'lishining zaruriy shartini ayting.
5. Lebeg teoremasini bayon eting.
6. Integrallanuvchi funksiyalarning sinflarini ayting.
7. Chegaralangan va cheksiz ko'p nuqtalarda uzilishga ega bo'lgan integrallanuvchi funksiyalarga misol keltiring.
8. Chegaralangan, ammo integrallanuvchi bo'lmagan funksiyaga misol keltiring.

Mustaqil yechish uchun mashqlar

1. Aniq integralning ta'rifidan foydalanib $\int_1^4 x^3 dx$ topilsin.

Javob: $\frac{255}{4}$.

Aniq integalning geometrik ma'nosiga asoslanib quyidagi integrallar topilsin.

2. $\int_0^4 \sqrt{16-x^2}dx$. Javob: 4π.

3. $\int_1^3 (2x-1)dx$. Javob: 6.

4. $\int_0^{\pi} \sin 2x\, dx$. Javob: 0.

5. $\int_{-2}^{2} x^5\, dx$. Javob: 0.

38. ANIQ INTEGRALNING ASOSIY XOSSALARI. ANIQ INTEGRALNI HISOBLASH

38.1. Aniq integralning asosiy xossalari

1-xossa. O'zgarmas ko'paytuvchini aniq integral belgisidan tashqariga chiqarish mumkin, ya'ni *A=const* bo'lsa

$$\int_a^b Af(x)dx = A\int_a^b f(x)dx$$

bo'ladi, bunda *f(x)* integrallashuvchi funksiya.

Isboti.

$$\int_a^b Af(x)dx = \lim_{\lambda\to 0}\sum_{k=1}^{n} Af(z_k)\Delta x_k = A\lim_{\lambda\to 0}\sum_{k=1}^{n} f(z_k)\Delta x_k = A\int_a^b f(x)dx.$$

2-xossa. Bir nechta (chekli sondagi) integrallashuvchi funksiyalarning algebraik yig'indisining aniq integrali qo'shiluvchilar aniq integrallarining yig'indisiga teng, ya'ni

$$\int_a^b [f(x)\pm\varphi(x)]dx = \int_a^b f(x)dx \pm \int_a^b \varphi(x)dx .$$

Isboti.

$$\int_a^b [f(x)\pm\varphi(x)]dx = \lim_{\lambda\to 0}\sum_{k=1}^{n}[f(z_k)\pm\varphi(z_k)]\Delta x_k = \lim_{\lambda\to 0}\left[\sum_{k=1}^{n} f(z_k)\Delta x_k \pm \sum_{k=1}^{n}\varphi(z_k)\Delta x_k\right] =$$

$$\lim_{\lambda\to 0}\sum_{k=1}^{n} f(z_k)\Delta x_k \pm \lim_{\lambda\to 0}\sum_{k=1}^{n}\varphi(z_k)\Delta x_k = \int_a^b f(x)dx \pm \int_a^b \varphi(x)dx.$$

3-xossa. Agar quyidagi uch integralning har biri mavjud bo'lsa, u holda har qanday uchta *a,b,c* sonlar uchun

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \qquad (38.1)$$

tenglik o'rinli bo'ladi.

Isboti. Dastlab $a<c<b$ deb faraz qilib *f(x)* funksiya uchun *[a,b]* kesmada integral yig'indi σ_n ni tuzamiz. Integral yig'indining limiti

[a,b] kesmani bo'laklarga bo'lish usuliga bog'liq bo'lmagani uchun *[a,b]* kesmani mayda kesmachalarga shunday bo'lamizki, *c* nuqta bo'lish nuqtasi bo'lsin.

Agar, masalan, $c=x_m$ bo'lsa, u holda σ_n integral yig'indini ikkita yig'indiga ajratamiz:

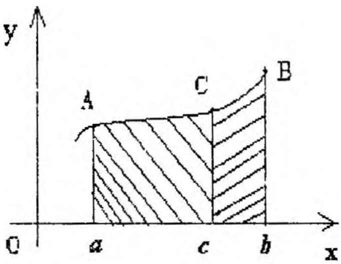
$$\sigma_n=\sum_{k=1}^{n} f(z_k)\Delta x_k=\sum_{k=1}^{m} f(z_k)\Delta x_k+\sum_{k=m+1}^{n} f(z_k)\Delta x_k .$$

Ushbu tenglikda $\lambda \to 0$ da limitga o'tsak isbotlanishi lozim bo'lgan (38.1) kelib chiqadi.

$a<b<c$ bo'lsin. U holda isbotlanganga muvofiq $\int_a^c f(x)dx=\int_a^b f(x)dx+\int_b^c f(x)dx$ bo'ladi.

Bundan $\int_a^b f(x)dx=\int_a^c f(x)dx-\int_b^c f(x)dx=\int_a^c f(x)dx+\int_c^b f(x)dx,$

ya'ni (38.1) ga ega bo'ldik.



148-chizma.

148-chizmada $f(x)>0$ va $a<c<b$ bo'lgan hol uchun 3-xossaning geometrik tasviri berilgan: *a A B b* egri chiziqli trapetsiyaning yuzi *a A C c* va *c C B b* egri chiziqli trapetsiyalar yuzlarini yig'indisiga teng.

4-xossa. Agar $[a,b]$ kesmada $f(x)$ funksiya integrallanuvchi va $f(x)\geq 0$ bo'lsa, u holda $\int_a^b f(x)dx \geq 0$ bo'ladi.

Isboti. Istalgan k uchun $f(x_k)\geq 0$, $\Delta x_k > 0$ bo'lgani sababli $\sum_{k=1}^{n} f(x_k)\Delta x_k \geq 0$ bo'ladi. Bunda $\lambda \to 0$ da limitga o'tsak isbotlanishi lozim bo'lgan tengsizlikni hosil qilamiz.

Shuningdek $[a,b]$ kesmada $f(x)$ funksiya integrallanuvchi va $f(x)\leq 0$ bo'lganda $\int_a^b f(x)dx \leq 0$ bo'lishini ko'rsatish qiyin emas.

5-xossa. Agar $[a,b]$ $(a<b)$ kesmada ikkita integrallanuvchi $f(x)$ va $\varphi(x)$ funksiya $f(x) \geq \varphi(x)$ shartni qanoatlantirsa,u holda

$$\int_a^b f(x)dx \geq \int_a^b \varphi(x)dx$$

tengsizlik o'rinli.

Isboti. $[a,b]$ da $f(x)-\varphi(x) \geq 0$ bo'lgani uchun 4- xossaga ko'ra $\int_a^b [f(x)-\varphi(x)]dx \geq 0$ bo'ladi. Bundan 2-xossasiga binoan

$$\int_a^b f(x)dx - \int_a^b \varphi(x)dx \geq 0 \quad \text{yoki} \quad \int_a^b f(x)dx \geq \int_a^b \varphi(x)dx$$

kelib chiqadi.

6- xossa. Agar $f(x)$ va $|f(x)|$ funksiya $[a,b]$ da integrallanuvchi bo'lsa, u holda

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx \qquad (38.2)$$

tengsizlik o'rinli.

Isboti. $-|f(x)|\leq f(x)\leq |f(x)|$ ga 5- xossani qo'llasak

$$-\int_a^b |f(x)|dx \leq \int_a^b f(x)dx \leq \int_a^b |f(x)|dx \quad \text{yoki} \quad \left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$$

tengsizlik hosil bo'ladi.

Natija. Agar *[a,b]* kesmada *f(x)* va $|f(x)|$ funksiya integrallanuvchi bo'lib, shu kesmada $|f(x)| \leq k$ *(k=const)* bo'lsa, u holda

$$\left|\int_a^b f(x)dx\right| \leq k(b-a) \qquad (38.3)$$

tengsizlik o'rinli.

Haqiqatdan, $|f(x)| \leq k$ bo'lgani uchun 6-5 va 1-xossaga asosan

$$\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx \leq k\int_a^b dx$$

bo'ladi. Bunda

$$\int_a^b dx = \lim_{\lambda \to 0} \sum_{k=1}^{n} 1 \cdot \Delta x_k = \Delta x_1 + \Delta x_2 + ... + \Delta x_n = b-a$$

ekanini hisobga olsak (38.3) tengsizlikka ega bo'lamiz.

7- xossa. (Aniq integralni baholash). Agar *m* va *M* sonlar *[a,b]* kesmada integrallanuvchi *f(x)* funksiyaning eng kichik va eng katta qiymati bo'lsa, u holda

$$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a) \qquad (38.4)$$

tengsizlik o'rinli.

Isboti. Shartga binoan *[a,b]* kesmadagi barcha *x* lar uchun $m \leq f(x) \leq M$.

Bunga 5- xossani qo'llasak

$m\int_a^b dx \leq \int_a^b f(x)dx \leq M\int_a^b dx$ yoki $\int_a^b dx = b-a$ ekanini hisobga olsak oxirgi tengsizliklardan (38.4) ga ega bo'lamiz

8- xossa. Agar *f(x)* funksiya *[a,b]* kesmada integrallanuvchi bo'lib *m* va *M* uning shu kesmadagi eng kichik va eng katta qiymati bo'lsa, u holda shunday o'zgarmas μ $(m \leq \mu \leq M)$ son mavjudki

$$\int_a^b f(x)dx = \mu \cdot (b-a) \quad (38.5)$$

tenglik o‘rinli.

Isboti. (38.4) ni *b-a* ga bo‘lsak $m \le \frac{1}{b-a}\int_a^b f(x)dx \le M$ bo‘ladi. $\frac{1}{b-a}\int_a^b f(x)dx = \mu$ belgilashni kiritamiz. U holda oxirgi tenglikni *b-a* ga ko‘paytirib isbotlanishi lozim bo‘lgan (38.5) tenglikka ega bo‘lamiz.

Natija (o‘rta qiymat haqidagi teorema). Agar *f(x)* *[a,b]* kesmada uzluksiz funksiya bo‘lsa, u holda kesmada shunday *x=c* nuqta topiladiki, bu nuqtada

$$\int_a^b f(x)dx = f(c)(b-a) \quad (38.6)$$

tenglik o‘rinli.

Haqiqatdan *f(x)* funksiya *[a,b]* kesmada uzluksiz bo‘lganligi tufayli u shu kesmada o‘zining eng kichik *m* va eng katta *M* qiymatini qabul qiladi. Uzluksiz funksiya *[m,M]* kesmadagi barcha qiymatlarni qabul qilganligi sababli u $\mu = \frac{1}{b-a}\int_a^b f(x)dx$ qiymatni ham qabul qiladi, ya‘ni *[a,b]* kesmada shunday *x=c* nuqta mavjud bo‘lib *f(c)*= μ bo‘ladi. (38.5) tenglikka μ o‘rniga *f(c)* ni qo‘yib isbotlanishi lozim bo‘lgan (38.6) tenglikni hosil qilamiz.

$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$ qiymat *f(x)* funksiyaning *[a,b]* kesmadagi **o‘rtacha qiymati** deb ataladi

Bu natijaga quyidagicha geometrik izoh berish mumkin. *[a,b]* kesmada $f(x) \geq 0$ bo‘lganda aniq integralning qiymati asosi *b-a* va balandligi *f(c)* bo‘lgan to‘g‘ri to‘rtburchakning yuziga teng bo‘lar ekan.

Agar $f(x)$ va $g(x)$ funksiyalar $[a,b]$ kesmada integrallanuvchi bo'lsa, u holda ularning ko'paytmasi $f(x)\cdot g(x)$ ham shu kesmada integrallashuvchi bo'lishini ta'kidlab o'tamiz.

38.2. Integralning yuqori chegarasi bo'yicha hosilasi

Agar aniq integralda integrallashning quyi chegarasi a ni aniq qilib belgilansa va yuqori chegarasi x esa o'zgaruvchi bo'lsa, u holda integralning qiymati ham x o'zgaruvchining funksiyasi bo'ladi.

Quyi chegarsi a o'zgarmas bo'lib yuqori chegarasi x o'zgaruvchi bo'lgan

$\int_a^x f(t)dt$ $(a \le x \le b)$ integralni qaraymiz. Bu integral yuqori chegara x ning funksiyasi bo'lganligi sababli uni $\phi(x)$ orqali belgilaymiz, ya'ni

$$\phi(x) = \int_a^x f(t)dt$$

va uni yuqori chegarsi o'zgaruvchi integral deb ataymiz.

38.1-teorema. Agar $f(x)$ funksiya $[a,b]$ kesmada uzluksiz bo'lsa, u holda

$$\phi'(x) = \left(\int_a^x f(t)dt \right)' = f(x)$$

tenglik o'rinli.

Isboti. $[a,b]$ ga tegishli istalgan x ni olib unga shunday $\Delta x \neq 0$ orttirma beramizki $x + \Delta x$ ham $[a,b]$ ga tegishli bo'lsin. U holda $\phi(x)$ funksiya

$$\phi(x + \Delta x) = \int_a^{x+\Delta x} f(t)dt$$

yangi qiymatni qabul qiladi. Aniq integralning 3-xossasiga ko'ra

$$\phi(x + \Delta x) = \int_a^{x+\Delta x} f(t)dt = \int_a^x f(t)dt + \int_x^{x+\Delta x} f(t)dt = \phi(x) + \int_x^{x+\Delta x} f(t)dt$$

bo'ladi. Demak, $\phi(x)$ funksiyaning orttirmasi

$$\Delta\phi(x) = \phi(x+\Delta x) - \phi(x) = \int_x^{x+\Delta x} f(t)dt$$

bo'ladi.

Oxirgi tenglikka o'rta qiymat haqidagi teoremani qo'llasak

$$\Delta\phi(x)=f(c)(x+\Delta x-x)=f(c)\Delta x$$

hosil bo'ladi, bunda c x bilan $x+\Delta x$ orasidagi son. Tenglikni har ikkala tomonini Δx ga bo'lamiz:

$$\frac{\Delta\phi(x)}{\Delta x} = f(c).$$

Agar Δx 0 ga intilsa c x ga intiladi va $f(x)$ funksiyaning $[a,b]$ kesmada uzluksizligidan $f(c)$ ning $f(x)$ ga intilishi kelib chiqadi.

Shuning uchun oxirgi tenglikda $\Delta x \to 0$ da limitga o'tib quyidagini hosil qilamiz:

$$\phi'(x) = \lim_{\Delta x\to 0}\frac{\Delta\phi(x)}{\Delta x} = \lim_{\Delta x\to 0}\frac{\phi(x+\Delta x)-\phi(x)}{\Delta x} = \lim_{\Delta x\to 0} f(c) = \lim_{c\to x} f(c) = f(x).$$

Bu teoremaga binoan $[a,b]$ kesmada uzluksiz $f(x)$ funksiya boshlang'ich funksiyaga ega ekanligi va $\phi(x) = \int_a^x f(t)dt$ shu funksiyaning boshlang'ich funksiyalaridan biri bo'lishi kelib chiqadi.

Agar $f(x)$ ning boshqa boshlang'ich funksiyalari uning $\phi(x)$ boshlang'ich funksiyasidan faqatgina o'zgarmas C songa farq qilishini hisobga olsak, aniqmas va aniq integrallar orasida bog'lanish o'rnatuvchi

$$\int f(x)dx = \int_a^x f(t)dt + C$$

tenglikka ega bo'lamiz.

38.3 Aniq integralni hisoblash. Nyuton-Leybnis formulasi

Aniq integrallarni integral yig'indining limiti sifatida bevosita hisoblash ko'p hollarda juda qiyin, uzoq hisoblashlarni talab qiladi va amalda juda kam qo'llaniladi. Aniq integralni hisoblash uchun Nyuton-Leybnis formulasini kashf etilishi aniq integralni qo'llanish ko'lamini kengayishiga asosiy sabab bo'ldi.

38.2-teorema. Agar *F(x)* funksiya uzluksiz *f(x)* funksiyaning *[a,b]* kesmadagi boshlang'ich funksiyasi bo'lsa, u holda $\int_a^b f(x)dx$ aniq integral boshlang'ich funksiyaning integrallash oralig'idagi orttirmasiga teng, ya'ni

$$\int_a^b f(x)dt = F(b) - F(a) \qquad (38.7).$$

(38.7) tenglik aniq integralni hisoblashning **aosiy formulasi** yoki **Nyuton-Leybnis formulasi** deyiladi.

Isboti. Shartga ko'ra *F(x)* funksiya *f(x)* ning biror boshlang'ich funksiyasi. $\phi(x) = \int_a^x f(t)dt$ funksiya ham *f(x)* ning boshlang'ich funksiyasi bo'lganligi uchun

$$\phi(x) = F(x) + C \quad \text{yoki} \quad \int_a^x f(t)dt = F(x) + C.$$

$$x = a \text{ desak } \int_a^a f(t)dt = F(a) + C, \; 0 = F(a) + C, \; C = -F(a).$$

Demak,

$$\int_a^x f(t)dt = F(x) - F(a).$$

Endi $x=b$ deb, Nyuton-Leybnis formulasini hosil qilamiz:

$$\int_a^b f(t)dt = F(b) - F(a).$$

$F(b) - F(a) = F(x)\Big|_a^b$ belgilash kiritilsa Nyuton-Leybnis formulasi

$$\int_a^b f(x)dx = F(x)\Big|_a^b \qquad (38.8)$$

ko'rinishga ega bo'ladi.

Demak, $[a,b]$ kesmada uzluksiz *f(x)* funksiyaning *F(x)* boshlang'ich funksiyasi elementar funksiya bo'lsa, u holda *f(x)* funksiyadan $[a,b]$ kesmada olingan aniq itegral (38.8) Nyuton-Leybnis formulasi yordamida hisoblanar ekan.

Boshlang'ich funksiyasi elementar funksiya bo'lmagan uzluksiz funksiyalardan olingan aniq integrallarni to'g'ridan-to'g'ri Nyuton-Leybnis formulasidan foydalanib hisoblab bo'lmaydi. Lekin ularning ba'zi birlarini har xil sun'iy usullarni qo'llash orqali hisoblash mumkin (5- va 6-misollarga qarang).

1-misol. Integralni hisoblang:

$$\int_0^{\frac{\pi}{2}} \sin x dx .$$

Yechish. *(-cosx)'=sinx* bo'lgani uchun

$$\int_0^{\frac{\pi}{2}} \sin x dx = -\cos x \Big|_0^{\frac{\pi}{2}} = -\left(\cos\frac{\pi}{2} - \cos 0\right) = -(0-1) = 1.$$

2-misol. $\int_a^b x^\alpha dx = \frac{x^{\alpha+1}}{\alpha+1}\Big|_a^b = \frac{b^{\alpha+1} - a^{\alpha+1}}{\alpha+1} \quad (\alpha \neq -1).$

3-misol. $\int\limits_{\frac{\pi}{6}}^{\frac{\pi}{4}} \frac{dx}{\sin^2 x} = -ctgx \Big|_{\frac{\pi}{6}}^{\frac{\pi}{4}} = -\left(ctg\frac{\pi}{4} - ctg\frac{\pi}{6}\right) = -(1-\sqrt{3}) = \sqrt{3}-1.$

Shunday qilib *[a,b]* kesmada uzluksiz *f(x)* funksiya uchun $\int f(x)dx = F(x)+C$ bo‘lganda $\int\limits_a^b f(x)dx = F(b)-F(a) = F(x)\Big|_a^b$ bo‘lar ekan.

Endi boshlang‘ich funksiya tuhunchasiga olib kelgan fizika masalasiga yana qaytamiz. Aytaylik to‘g‘ri chiziqli bir tomonlama harakatda moddiy nuqtaning *v=v(t)* tezligini bilgan holda [*0,T*] vaqt oralig‘ida uning bosib o‘tgan yo‘lini topish talab etilsin.

Moddiy nuqta *0x* o‘q bo‘ylab *x=x(t)* harakat tenglamasiga binoan harakatlanadi deb faraz qilsak, u holda hosilaning mexanik ma’nosiga ko‘ra

$$v(t) = \frac{dx}{dt}$$

bo‘lar edi, bundan *dx=v(t)dt*.

Buni *0* dan *T* gacha integrallab moddiy nuqtaning *T* vaqt ichida bosib o‘tgan yo‘lini topamiz:

$$S = x(T) - x(0) = \int\limits_0^T v(t)dt.$$

Oxirgi tenglikdan

$$x = x_0 + \int\limits_0^T v(t)dt$$

moddiy nuqtaning harakat tenglamasiga ega bo‘lamiz, bunda $x_0 = x(0)$.

Demak, to‘g‘ri chiziqli bir tomonlama harakatda [*0,T*] vaqt oralig‘ida moddiy nuqtaning bosib o‘tgan yo‘li [*0,T*] kesmada *v(t)* tezlikdan olingan aniq integralga teng ekan. Bu aniq integralning fizik (mexanik) ma’nosidir.

38.4. Aniq integralda o'zgaruvchini almashtirish

$$\int_a^b f(x)dx$$

integralni hisoblash talab etilsin, bunda $f(x)$ funksiya $[a,b]$ kesmada uzluksiz. $x=\varphi(t)$ almashtirish olamiz, bunda $\varphi(t)$ $[\alpha,\beta]$ kesmada uzluksiz va uzluksiz $\varphi'(t)$ hosilaga ega hamda $\varphi(\alpha)=a$, $\varphi(\beta)=b$ bo'lsin. U holda

$$\int_a^b f(x)dx=\int_\alpha^\beta f(\varphi(t))\varphi'(t)dt$$

formula o'rinli bo'ladi.

Haqiqatan, agar $F(x)$ funksiya $f(x)$ ning boshlang'ich funksiyasi bo'lsa, u holda $F(\varphi(t))$ funksiya $f(\varphi(t))\,\varphi'(t)$ funksiya uchun boshlang'ich funksiya bo'lishi isbotlangan edi. Nyuton-Leybnis formulasiga ko'ra

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b)-F(a);$$

$$\int_\alpha^\beta f(\varphi(t))\varphi'(t)dt = F(\varphi(t))\Big|_\alpha^\beta = F(\varphi(\beta))-F(\varphi(\alpha)) = F(b)-F(a).$$

4-misol. $\int_0^1 \sqrt{1-x^2}dx$ hisoblansin.

Yechish. $x=\sin t$ deb almashtirsak, $dx=\cos t dt$, $1-x^2=\cos^2 t$ bo'ladi. $x=0$ da $\sin t=0$, $t=0$, $x=1$ da $\sin t=1$, $t=\frac{\pi}{2}$.

$$\int_0^1 \sqrt{1-x^2}dx = \int_0^{\frac{\pi}{2}} \cos^2 t dt = \int_0^{\frac{\pi}{2}} \frac{1+\cos 2t}{2}dt = \frac{1}{2}\int_0^{\frac{\pi}{2}}(1+\cos 2t)dt = \frac{1}{2}\left(t+\frac{\sin 2t}{2}\right)\Big|_0^{\frac{\pi}{2}} =$$

$$= \frac{1}{2}\left[\frac{\pi}{2}+\frac{\sin\pi}{2}-\left(0+\frac{\sin 0}{2}\right)\right] = \frac{\pi}{4}.$$

5-misol. Ushbu $J=\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx$ integral hisoblansin.

Yechish. Berilgan integralni ikkita integrallarni yig'indisi ko'rinishda tasvirlaymiz:

$$J=\int_0^{\frac{\pi}{2}}\frac{x\sin x}{1+\cos^2 x}dx+\int_{\frac{\pi}{2}}^{\pi}\frac{x\sin x}{1+\cos^2 x}dx=J_1+J_2.$$

J_2 integralda $x=\pi-t,\ dx=-dt$ almashtirish olamiz. U holda $x=\frac{\pi}{2}$ da $t=\frac{\pi}{2}$ va $x=\pi$ da $t=0$ bo'lgani uchun

$$J_2=-\int_{\frac{\pi}{2}}^{0}\frac{(\pi-t)\sin(\pi-t)}{1+\cos^2(\pi-t)}dt=\int_0^{\frac{\pi}{2}}\frac{(\pi-t)\sin t}{1+\cos^2 t}dt=\pi\int_0^{\frac{\pi}{2}}\frac{\sin tdt}{1+\cos^2 t}-\int_0^{\frac{\pi}{2}}\frac{t\sin t}{1+\cos^2 t}dt$$

Demak,

$$J=J_1+J_2=\int_0^{\frac{\pi}{2}}\frac{x\sin x}{1+\cos^2 x}dx+\pi\int_0^{\frac{\pi}{2}}\frac{\sin tdt}{1+\cos^2 t}-\int_0^{\frac{\pi}{2}}\frac{t\sin t}{1+\cos^2 t}dt.$$

Birinchi va uchinchi integrallar faqatgina integrallash o'zgaruvchisini belgilanishi bilan farq qilgani uchun

$$J=\pi\int_0^{\frac{\pi}{2}}\frac{\sin tdt}{1+\cos^2 t}$$

bo'ladi.

Bu integralda $\cos t=u,\ du=-\sin tdt$ almashtirish olsak $t=0$ da $u=1$, $t=\pi/2$ da $u=0$ bo'lib

$$J=-\pi\int_1^0\frac{du}{1+u^2}=\pi\int_0^1\frac{du}{1+u^2}=\pi\cdot\operatorname{arctg}u\Big|_0^1=$$

$$=\pi(\operatorname{arctg}1-\operatorname{arctg}0)=\pi\left(\frac{\pi}{4}-0\right)=\frac{\pi^2}{4}$$

bo'ladi.

Izoh. $\int\frac{x\sin x}{1+\cos^2 x}dx$ aniqmas integral elementar funksiya orqali ifodalanmaydi. Lekin, berilgan aniq integralni sun'iy usul bilan hisoblash mumkinligini ko'rsatdik.

6-misol. $J=\int_0^1\frac{\ln(1+x)}{1+x^2}dx$ integral hisoblansin.

Yechish. $\int\frac{\ln(1+x)}{1+x^2}dx$ aniqmas integral elementar funksiyalar orqali ifodalanmaydi. Demak berilgan integralni bevosita Nyuton-Leybnis formulasi yordamida hisoblab bo'lmaydi.

$x=tgt,\ dx=\frac{dt}{\cos^2 t}$ almashtirish olamiz. U holda *x=0* da *t=0*, *x=1* da *t=π/4* bo'lgani uchun berilgan integral

$$J=\int_0^{\frac{\pi}{4}}\frac{\ln(1+tgt)\frac{1}{\cos^2 t}}{1+tg^2 t}dt=\int_0^{\frac{\pi}{4}}\ln(1+tgt)dt$$

ko'rinishni oladi.

$$1+tgt=1+\frac{\sin t}{\cos t}=\frac{\cos t+\sin t}{\cos t}=\frac{\sqrt{2}\left(\frac{\sqrt{2}}{2}\cos t+\frac{\sqrt{2}}{2}\sin t\right)}{\cos t}=$$

$$=\frac{\sqrt{2}\left(\sin\frac{\pi}{4}\cos t+\cos\frac{\pi}{4}\sin t\right)}{\cos t}=\frac{\sqrt{2}\sin\left(\frac{\pi}{4}+t\right)}{\cos t},$$

$$\ln(1+tgt)=\ln\frac{\sqrt{2}\sin(\frac{\pi}{4}+t)}{\cos t}=\ln\sqrt{2}\sin(\frac{\pi}{4}+t)-\ln\cos t=$$

$$=\ln 2^{\frac{1}{2}}+\ln\sin(\frac{\pi}{4}+t)-\ln\cos t$$

ekanini hisobga olib

$$J=\int_0^{\frac{\pi}{4}}\frac{1}{2}\ln 2dt+\int_0^{\frac{\pi}{4}}\ln\sin(\frac{\pi}{4}+t)dt-\int_0^{\frac{\pi}{4}}\ln\cos tdt$$

ga va $\int_0^{\frac{\pi}{4}}\frac{1}{2}\ln 2dt=\frac{1}{2}\ln 2\cdot t\Big|_0^{\frac{\pi}{4}}=\frac{1}{2}\ln 2\cdot\frac{\pi}{4}=\frac{\pi}{8}\ln 2$

bo'lgani uchun

$$J=\frac{\pi}{8}\ln 2+\int_0^{\frac{\pi}{4}}\ln\sin(\frac{\pi}{4}+t)dt-\int_0^{\frac{\pi}{4}}\ln\cos tdt=\frac{\pi}{8}\ln 2+J_1-J_2$$

kelib chiqadi.

Endi $J_1=J_2$ ekanini ko'rsatamiz. Buning uchun $J_2=\int_0^{\frac{\pi}{4}}\ln\cos tdt$ integralda $t=\frac{\pi}{4}-z,\ dt=-dz$ almashtirish olamiz. U holda $t=0$ da $z=\frac{\pi}{4}$, $t=\frac{\pi}{4}$ da $z=0$ bo'lib

$$J_2=-\int_{\frac{\pi}{4}}^{0}\ln\cos(\frac{\pi}{4}-z)dz=\int_0^{\frac{\pi}{4}}\ln\sin[\frac{\pi}{2}-(\frac{\pi}{4}-z)]dz=\int_0^{\frac{\pi}{4}}\ln\sin(\frac{\pi}{4}+z)dz=J_1$$

tenglik hosil bo'ladi.

Shuning uchun $J=\frac{\pi}{8}\ln 2$.

Bundan buyn xuddi aniqmas integraldagi kabi aniq integralda ham o'zgaruvchini almashtirish jarayonini integraldan so'ng vertikal kesmalar orasiga yozamiz.

38.5. Aniq igtegralni bo'laklab integrallash

Faraz qilaylik. *u(x)* va *v(x)* funksiyalar *[a;b]* kesmada differensiallanuvchi funksiyalar bo'lsin. U holda

$$d(u\upsilon)'=\upsilon du+ud\upsilon'$$

bo'ladi, buni a dan b gacha integrallasak

$$\int_a^b d(uv)=\int_a^b vdu+\int_a^b udv, (uv)\Big|_a^b=\int_a^b vdu+\int_a^b udv,$$

bundan $\int_a^b udv=(uv)\Big|_a^b-\int_a^b vdu.$

Bu formula aniq integralni **bo'laklab integrallash** formulasi deyiladi.

Aniq integralni bo'laklab untegrallash jarayoniga aloqador belgilashlarni ham xuddi o'zgaruvchini almashtirish usulidagidek integraldan so'ng vertikal kesmalar orasiga yozamiz.

7- misol. $\int_1^e \ln xdx$ hisoblansin.

Yechish.

$$\int_1^e \ln xdx\left|\begin{matrix} u=\ln x, dv=dx \\ du=(\ln x)'dx=\dfrac{1}{x}dx, v=x\end{matrix}\right|=\ln x\cdot x\Big|_1^e-\int_1^e x\frac{dx}{x}=$$

$$=\ln e\cdot e-\ln 1\cdot 1-x\Big|_1^e=e-(e-1)=e-e+1=1$$

8-misol.

$$\int_0^2 xe^{-x}dx\left|\begin{aligned}&u=x, dv=e^{-x}dx\\&du=dx, v=\int e^{-x}dx=-e^{-x}\end{aligned}\right|=x\cdot(-e^{-x})\Big|_0^2+\int_0^2 e^{-x}dx=$$

$$=-(2e^{-2}-0\cdot e^{-0})-e^{-x}\Big|_0^2=-\frac{2}{e^2}-(e^{-2}-e^{-0})=-\frac{2}{e^2}-\frac{1}{e^2}+1=1-\frac{3}{e^2}.$$

9-misol.

$$\int_0^{\frac{\pi}{2}} x\cos xdx\left|\begin{aligned}&u=x, \cos xdx=dv,\\&du=dx, v=\int\cos xdx=\sin x\end{aligned}\right|=x\cdot\sin x\Big|_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}\sin xdx=$$

$$=\frac{\pi}{2}\cdot\sin\frac{\pi}{2}-0\cdot\sin 0+\cos x\Big|_0^{\frac{\pi}{2}}=\frac{\pi}{2}-0+\cos\frac{\pi}{2}-\cos 0=\frac{\pi}{2}-1.$$

10-misol.

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\frac{xdx}{\sin^2 x}\left|\begin{aligned}&u=x, dv=\frac{dx}{\sin^2 x}\\&du=dx, v=\int\frac{dx}{\sin^2 x}=-\text{ctg}x\end{aligned}\right|=x\cdot(-\text{ctg}x)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}-\int_{\frac{\pi}{4}}^{\frac{\pi}{3}}(-ctgx)dx=-\left(\frac{\pi}{3}\cdot\text{ctg}\frac{\pi}{3}-\frac{\pi}{4}\text{ctg}\frac{\pi}{4}\right)+$$

$$+\ln\sin x\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}=-\left(\frac{\pi}{3}\cdot\frac{1}{\sqrt{3}}-\frac{\pi}{4}\cdot 1\right)+\ln\sin\frac{\pi}{3}-\ln\sin\frac{\pi}{4}=-\frac{\pi}{3\sqrt{3}}+\frac{\pi}{4}+\ln\frac{\sqrt{3}}{2}-\ln\frac{\sqrt{2}}{2}=$$

$$=\frac{\pi}{4}-\frac{\pi}{3\sqrt{3}}+\ln\left(\frac{\sqrt{3}}{2}:\frac{\sqrt{2}}{2}\right)=\frac{\pi}{4}-\frac{\sqrt{3}\pi}{9}+\ln\sqrt{\frac{3}{2}}=\frac{9\pi-4\sqrt{3}\pi}{36}+\frac{1}{2}\ln\frac{3}{2}.$$

O‘z-o‘zini tekshirish uchun savollar

1. Aniq integralning xossalarini ayting.
2. Aniq integralni baholang.
3. O‘rta qiymat haqidagi teoremani isbotlang.
4. Funksiyaning kesmadagi o‘rtacha qiymatini ayting.
5. Integralning yuqori chegarasi bo‘yicha hosilasi nimaga teng?

6. Uzluksiz funksiyaning boshlang‘ich funksiyasi mavjudligini isbotlang.

7. Nyuton-Leybnis formulasini isbotlang.

8. Aniq integralda o‘zgaruvchini almashtirish formulasini isbotlang.

9. Aniq integralni bo‘laklab integrallash formulasini isbotlang.

10. Bo‘laklab integrallashda muhim o‘rinda nima turadi?

Mustaqil yechish uchun mashqlar

Funksiyalarning hosilasini toping.

1. $F(x)=\int_2^x e^{-3t}dt, x>2.$ Javob: e^{-3x}.

2. $F(x)=\int_{x^3}^0 \sin 2t dt$; $x>0$ Javob: $-3x^2 \sin 2x^3$.

3. $f(x)=3-2\sin x$ funksiyaning $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ kesmadagi o‘rtacha qiymatini toping. Javob: 3.

Integrallar baholansin.

4. $I=\int_0^2 \sqrt{8-x^2}dx$. Javob: $4<I<4\sqrt{2}$.

5. $I=\int_0^\pi \frac{dx}{5-3\cos x}$. Javob: $\frac{\pi}{8}<I<\frac{\pi}{2}$.

Integrallarni Nyuton-Leybnis formulasidan foydalanib hisoblang.

6. $\int_{-1}^1 (6x^2-2x-5)dx$. Javob: -6.

7. $\int_0^1 \frac{dx}{(5-3x)^3}$. Javob: $\frac{7}{200}$.

8. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x \cdot \cos 3x dx$. Javob: *0.*

9. $\int_{3}^{4} \frac{dx}{25-x^2}$. Javob: $\frac{1}{5}\ell n\frac{3}{2}$.

10. $\int_{1}^{4} \frac{dx}{x^2-2x+10}$. Javob: $\frac{\pi}{12}$.

Integrallarni o'zgaruvchini almashtirib hisoblang.

11. $\int_{3}^{8} \frac{dx}{5-\sqrt{x+1}}$. Javob: $10\ln\frac{3}{2}-2$.

12. $\int_{0}^{3} \frac{dx}{\sqrt{(16+x^2)^3}}$. Javob: $\frac{3}{80}$.

13. $\int_{\frac{\sqrt{2}}{2}}^{1} \frac{\sqrt{1-x^2}}{x^2}dx$. Javob: $1-\frac{\pi}{4}$.

14. $\int_{0}^{\frac{\pi}{2}} \frac{dx}{5+4\cos x}$. Javob: $\frac{2}{3}\operatorname{arctg}\frac{1}{3}$.

Integrallarni bo'laklab integrallang.

15. $\int_{0}^{2} \frac{xdx}{e^{2x}}$. Javob: $\frac{e^4-5}{4e^4}$.

16. $\int_{0}^{\pi} x\sin\frac{x}{2}dx$. Javob: *4.*

17. $\int_{1}^{\sqrt{e}} x\ln x dx$.. Javob: $\frac{e+1}{4}$.

18. $\int_{0}^{\frac{\pi}{3}} \frac{xdx}{\cos^2 x}$. Javob: $\frac{\pi}{3}-\ln 2$.

39. XOSMAS INTEGRALLAR

39.1. Aniq integral tushunchasini umumlashtirish

Aniq integralga ta'rif berishda integrallash oralig'i $[a,b]$ chekli va integral ostidagi funksiya shu oraliqda chegaralangan deb faraz qilgan edik. Ana shu shartlardan aqalli birortasi bajarilmasa aniq integralning keltirilgan ta'rifi ma'nosini yo'qotadi. Chunki integrallash oralig'i cheksiz bo'lganda uni uzunliklari chekli bo'lgan n ta qismga ajratib bo'lmaydi, integral ostidagi funksiya chegaralanmaganda integral yig'indi chekli limitga ega bo'lmaydi. Ammo aniq integral tushunchasini bu hollar uchun ham umumlashtirish mumkin. Umumlashtirish natijasida xosmas integrallar tushunchasiga kelamiz. Xosmas integrallar ikki turga- chegaralari cheksiz xosmas integrallar hamda chegaralanmagan funksyaning xosmas integraliga bo'linadi.

39.2. Chegaralari cheksiz xosmas integrallar

1-ta'rif. $f(x)$ funksiya $[a,\infty)$ intervalda aniqlangan bo'lib, u istalgan chekli $[a,R]$ $(R>a)$ kesmada integrallanuvchi, ya'ni $\int_a^R f(x)dx$ aniq integral mavjud bo'lsin. U holda

$$\lim_{R\to+\infty}\int_a^R f(x)dx \qquad (39.1)$$

chekli limit mavjud bo'lsa, u **birinchi tur** yoki **chegaralari cheksiz** xosmas integral deb ataladi, va

$$\int_a^{+\infty} f(x)dx \qquad (39.2)$$

kabi belgilanadi.

Shunday qilib, ta'rifga ko'ra

$$\int_{a}^{+\infty} f(x)dx = \lim_{R\to+\infty}\int_{a}^{R} f(x)dx.$$

Bu holda (39.2.) xosmas integral **mavjud** yoki **yaqinlashadi** deyiladi. Agar (39.1) limit mavjud bo‘lmasa, u holda (39.2) xosmas integral **mavjud emas** yoki **uzoqlashadi** deyiladi.

(-∞,b] intervalda chegaralangan *f(x)* funksiyaning xosmas integrali ham (39.2) kabi aniqlanadi:

$$\int_{-\infty}^{b} f(x)dx = \lim_{R\to-\infty}\int_{R}^{b} f(x)dx. \qquad (39.3)$$

Bunda *f(x)* funksiya istalgan *[R,b]* *(R<b)* kesmada integrallanuvchi.

Shuningdek,

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{+\infty} f(x)dx \qquad (39.4)$$

tenglik yordamida *f(x)* funksiyaning *(-∞,+∞)* bo‘yicha xosmas integrali aniqlanadi. Bunda *c* ixtiyoriy o‘zgarmas son. (39.4) tenglikning o‘ng tomonidagi har ikkala xosmas integrallar yaqinlashgada uning chap tomonidagi xosmas interal ham yaqinlashadi.

Endi birinchi tur xosmas integralni hisoblashga misollar keltiramiz.

1- misol. $\int_{0}^{+\infty}\frac{dx}{1+x^2} = \lim_{R\to\infty}\int_{0}^{R}\frac{dx}{1+x^2} = \lim_{R\to\infty}\operatorname{arctg}x\Big|_{0}^{R} = \lim_{R\to\infty}\operatorname{arctg}R = \frac{\pi}{2}$,

ya‘ni berilgan xosmas integral yaqinlashadi.

2- misol.

$$\int_{0}^{+\infty}\sin x dx = \lim_{R\to+\infty}\int_{0}^{R}\sin x dx = \lim_{R\to+\infty}(-\cos x)\Big|_{0}^{R} = \lim_{R\to+\infty}(-\cos R+1) =$$

$$= 1 - \lim_{R\to+\infty}\cos R,$$

ammo cos*R* funksiya *R*→+∞ da limitga ega bo‘lmaganligi uchun integral uzoqlashadi.

3-misol. $\int_1^{+\infty}\frac{dx}{x^\alpha}$, α-biror son.

Shu birinchi tur xosmas integralni α ning qanday qiymatlarida yaqinlashishi va qanday qiymatlarida uzoqlashishini aniqlaymiz.

a) $\alpha \neq 1$ bo'lsin, u holda istalgan $R>0$ uchun

$$\int_1^{+\infty}\frac{dx}{x^\alpha}=\lim_{R\to+\infty}\int_1^R\frac{dx}{x^\alpha}=\lim_{R\to+\infty}\int_1^R x^{-\alpha}dx=\lim_{R\to+\infty}\frac{x^{-\alpha+1}}{-\alpha+1}\Big|_1^R=$$

$$=\lim_{R\to+\infty}\frac{R^{1-\alpha}-1}{1-\alpha}=\begin{cases}\frac{1}{\alpha-1}, \text{agar } \alpha>1, \text{bo' lsa},\\ +\infty, \text{agar } \alpha< \text{bo' lsa}.\end{cases}$$

b) $\alpha=1$ bo'lsin, u holda istalgan $R>0$ son uchun

$$\int_1^{+\infty}\frac{dx}{x}=\lim_{R\to+\infty}\int_1^R\frac{dx}{x}=\lim_{R\to+\infty}\ln x\Big|_1^R=\lim_{R\to+\infty}\ln R=+\infty.$$

Shunday qilib, berilgan integral $\alpha>1$ bo'lganda yaqinlashadi, $\alpha\leq 1$ bo'lganda esa uzoqlashadi.

39.3. Chegaralanmagan funksiyaning xosmas integrali

2-ta'rif. $f(x)$ funksiya $[a,b)$ oraliqda aniqlangan bo'lsin. Agar $f(x)$ funksiya $x=b$ nuqtaning biror atrofida chegaralanmagan bo'lib, u $[a,b)$ ga tegishli har qanday $[a,b-\varepsilon]$ kesmada chegaralangan bo'lsa $x=b$ nuqta $f(x)$ ning **maxsus** nuqtasi deyiladi. $x=b$ nuqta $f(x)$ning maxsus nuqtasi bo'lib $f(x)$ funksiya istalgan $[a,b-\varepsilon]$ $(\varepsilon>0, b-\varepsilon>a)$ kesmada integrallashuvchi, ya'ni $\int_a^{b-\varepsilon}f(x)dx$ aniq integral mavjud bo'lsin. U holda

$$\lim_{\varepsilon\to+0}\int_a^{b-\varepsilon}f(x)dx \qquad (39.5)$$

chekli limit mavjud bo'lsa, uni **ikkinchi tur** yoki **chegaralanmagan funksiyaning** xos integrali deb ataladi va

$$\int_a^b f(x)dx \qquad (39.6)$$

ko'rinishda belgilanadi. Bu holda (39.6) integral mavjud yoki **yaqinlashadi** deb aytiladi. (39.5) limit mavjud bo'lmasa, u holda (39.6) integral mavjud emas yoki **uzoqlashadi** deb aytiladi.

Shuningdek $x=a$ nuqta $f(x)$ funksiyaning maxsus nuqtasi (a nuqtaning yaqin atrofida $f(x)$ chegaralanmagan va istalgan $[a+\varepsilon,b]$ $(\varepsilon>0, a+\varepsilon<b)$ kesmada chegaralangan) bo'lganda xosmas integral

$$\int_a^b f(x)dx = \lim_{\varepsilon\to+0}\int_{a+\varepsilon}^b f(x)dx$$

kabi aniqlanadi.

Agar $f(x)$ funksiya $[a,b]$ kesmaning biror ichki c nuqtasining qandaydir atrofida chegaralanmaganda xosmas integral

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

tenglik yordamida aniqlanadi. Bu tenglikning o'ng tomonidagi har ikkala integral yaqinlashganda uning chap tomonidagi xosmas integral ham yaqinlashadi.

Shuningdek, a va b nuqtalar $f(x)$ funksiyaning maxsus nuqtalari bo'lganda xosmas integral

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx \qquad (39.7)$$

tenglik yordamida aniqlanadi, bunda c (a,b) intervalining ixtiyoriy nuqtasi.

Izoh. Uzluksiz funksiyaning ikkinchi tur uzilish nuqtasi uning maxsus nuqtasi bo'ladi.

4- misol. $\int_a^b \frac{dx}{(x-a)^\alpha}$ integral tekshirilsin, bunda $\alpha>0$ - biror son.

Yechish. Integral ostidagi $\frac{1}{(x-a)^{\alpha}}$ funksiya uchun $x=a$ maxsus nuqta bo'ladi.

a) $\alpha \neq 1$ bo'lsin, u holda

$$\int_a^b \frac{dx}{(x-a)^{\alpha}} = \lim_{\varepsilon\to+0}\int_{a+\varepsilon}^b \frac{dx}{(x-a)^{\alpha}} = \lim_{\varepsilon\to+0}\int_{a+\varepsilon}^b (x-a)^{-\alpha}dx = \lim_{\varepsilon\to+0}\frac{(x-a)^{-\alpha+1}}{-\alpha+1}\Bigg|_{a+\varepsilon}^b =$$

$$=\begin{cases}\frac{(b-a)^{1-\alpha}}{1-\alpha}, \text{agar}\ \alpha<1 \quad \text{bo'lsa,}\\ +\infty, \text{agar}\ \alpha>1 \quad \text{bo'lsa.}\end{cases}$$

b) $\alpha=1$ bo'lsin, u holda

$$\int_a^b \frac{dx}{x-a} = \lim_{\varepsilon\to+0}\int_{a+\varepsilon}^b \frac{dx}{x-a} = \lim_{\varepsilon\to+0}\ln|x-a|\Bigg|_{a+\varepsilon}^b = \lim_{\varepsilon\to+0}\left[\ln(b-a)-\ln\varepsilon\right] = +\infty.$$

Shunday qilib berilgan integral $0<\alpha<1$ bo'lganda yaqinlashadi, $\alpha\geq 1$ bo'lganda esa uzoqlashadi.

Shuningdek $\int_a^b \frac{dx}{(b-x)^{\alpha}}$ integral ham $0<\alpha<1$ bo'lganda yaqinlashadi, $\alpha\geq 1$ bo'lganda esa uzoqlashadi.

5- misol. $\int_2^5 \frac{dx}{\sqrt[3]{x-3}}$ integral tekshirilsin.

Yechish. Integral ostidagi $\frac{1}{\sqrt[3]{x-3}}$ funksiya *[2,5]* kesmaning ichidagi $x=3$ nuqtada ikkinchi tur uzilishga ega, ya'ni bu nuqta funksyaining maxsus nuqtasi.

Ta'rifga binoan:

$$\int_2^5 \frac{dx}{\sqrt[3]{x-3}} = \int_2^3 \frac{dx}{\sqrt[3]{x-3}} + \int_3^5 \frac{dx}{\sqrt[3]{x-3}} = \lim_{\varepsilon_1 \to +0} \int_2^{3-\varepsilon_1} \frac{dx}{(x-3)^{\frac{1}{3}}} + \lim_{\varepsilon_2 \to +0} \int_{3+\varepsilon_2}^5 \frac{dx}{(x-3)^{\frac{1}{3}}} =$$

$$= \lim_{\varepsilon_1 \to +0} \frac{(x-3)^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} \Big|_2^{3-\varepsilon_1} + \lim_{\varepsilon_2 \to +0} \frac{(x-3)^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} \Big|_{3+\varepsilon_2}^5 = \lim_{\varepsilon_1 \to +0} \frac{3}{2} \sqrt[3]{(x-3)^2} \Big|_2^{3-\varepsilon_1} +$$

$$+ \lim_{\varepsilon_2 \to +0} \frac{3}{2} \sqrt[3]{(x-3)^2} \Big|_{3+\varepsilon_2}^5 = \frac{3}{2} \lim_{\varepsilon_1 \to +0} \left[\sqrt[3]{\varepsilon_1^2} - 1 \right] + \frac{3}{2} \lim_{\varepsilon_2 \to +0} \left[\sqrt[3]{2^2} - \sqrt[3]{\varepsilon_2^2} \right] = -\frac{3}{2} + \frac{3}{2}\sqrt[3]{4} = \frac{3}{2}\left(\sqrt[3]{4} - 1\right).$$

Demak, berilgan integral yaqinlashadi.

6-misol. $\int_0^1 \frac{dx}{\sqrt{x(1-x)}}$ tekshirilsin.

Yechish. Integral ostidagi $\frac{1}{\sqrt{x(1-x)}}$ funksiya uchun *a=0, b=1* nuqtalar maxsus nuqtalar bo‘ladi. Xosmas itntegralning ta‘rifidan foydalanib topamiz:

$$\int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{x(1-x)}} + \int_{\frac{1}{2}}^1 \frac{dx}{\sqrt{x(1-x)}} = \lim_{\varepsilon_1 \to +0} \int_{\varepsilon_1}^{\frac{1}{2}} \frac{dx}{\sqrt{x(1-x)}} + \lim_{\varepsilon_2 \to +0} \int_{\frac{1}{2}}^{1-\varepsilon_2} \frac{dx}{\sqrt{x(1-x)}} =$$

$$= \lim_{\varepsilon_1 \to +0} \int_{\varepsilon_1}^{\frac{1}{2}} \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2}} + \lim_{\varepsilon_2 \to +0} \int_{\frac{1}{2}}^{1-\varepsilon_2} \frac{dx}{\sqrt{\left(\frac{1}{2}\right)^2 - \left(x - \frac{1}{2}\right)^2}} = \lim_{\varepsilon_1 \to +0} \arcsin \frac{x - \frac{1}{2}}{\frac{1}{2}} \Big|_{\varepsilon_1}^{\frac{1}{2}} +$$

$$+ \lim_{\varepsilon_2 \to +0} \arcsin \frac{x - \frac{1}{2}}{\frac{1}{2}} \Big|_{\frac{1}{2}}^{1-\varepsilon_2} = \lim_{\varepsilon_1 \to +0} \left[\arcsin 0 - \arcsin(2\varepsilon_1 - 1) \right] -$$

$$- \lim_{\varepsilon_2 \to +0} \left[\arcsin(2\varepsilon_2 - 1) - \arcsin 0 \right] = \arcsin 1 + \arcsin 1 = 2 \cdot \frac{\pi}{2} = \pi.$$

Demak integral yaqinlashuvchi va

$$\int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \pi .$$

7- misol. $\int\limits_{-1}^{1}\frac{dx}{x^2}$ integral hisoblansin.

Yechish. Birinchi qarashda berilgan integral juda oson hisoblanadi, ya'ni

$$\int\limits_{-1}^{1}\frac{dx}{x^2}=-\frac{1}{x}\Big|_{-1}^{1}=-\left(\frac{1}{1}-\frac{1}{-1}\right)=-2.$$

Ammo olingan natija noto'g'ri. Bu noto'g'ri natijaga e'tiborsizligimiz oqibatida, ya'ni integral ostidagi $\frac{1}{x^2}$ funksiya *[-1,1]* kesmada $x=0$ maxsus nuqtaga ega ekanligini hisobga olmaganligimiz sababli keldik. Biz berilgan integralni oddiy aniq integral deb emas, balki xosmas integral deb qarashimiz lozim. Uni ta'rifdan foydalanib hisoblaymiz:

$$\int\limits_{-1}^{1}\frac{dx}{x^2}=\int\limits_{-1}^{0}\frac{dx}{x^2}+\int\limits_{0}^{1}\frac{dx}{x^2}=\lim_{\varepsilon_1\to+0}\int\limits_{-1}^{\varepsilon_1}\frac{dx}{x^2}+\lim_{\varepsilon_2\to+0}\int\limits_{\varepsilon_2}^{1}\frac{dx}{x^2}. \qquad (39.8)$$

Limitlardan birini hisoblaymiz:

$$\lim_{\varepsilon_1\to+0}\int\limits_{-1}^{\varepsilon_1}\frac{dx}{x^2}=\lim_{\varepsilon_1\to+0}\left(-\frac{1}{x}\right)\Big|_{-1}^{\varepsilon}=\lim_{\varepsilon\to+0}\left(+\frac{1}{\varepsilon}-\frac{1}{-1}\right)=\infty.$$

(39.8) tenglikning o'ng tomonidagi xosmas integrallardan biri uzoqlashganligi uchun ta'rifga binoan uning chap tomonidagi xosmas integral ham uzoqlashadi.

39.4. Xosmas integrallarning yaqinlashish alomatlari

Ko'p hollarda xosmas integralning qiymatini topish talab etilmasdan uning yaqinlashuvchi yoki uzoqlashuvchi ekanini bilishning o'zi kifoya qiladi. Bunday hollarda **taqqoslash teoremalari** deb ataluvchi quyidagi teoremalardan foydalanish mumkin.

39.1-teorema. Agar $f(x)$ va $\varphi(x)$ funksiyalar $[a, +\infty)$ oraliqda uzluksiz bo'lib, $0\leq f(x)\leq\varphi(x)$ shartni qanoatlantirsa, u holda *a)* $\int_a^{+\infty} \varphi(x)dx$ xosmas integral yaqinlashsa, $\int_a^{+\infty} f(x)dx$ integral ham yaqinlashadi.

b) $\int_a^{+\infty} f(x)dx$ integrla uzoqlashganda $\int_a^{+\infty} \varphi(x)dx$ integral ham uzoqlashadi.

Bu teorema faqatgina nomanfiy funksiyalarga tegishli bo'lib undan ishorasini saqlamaydigan funksiyalarning xosmas integrallarini tekshirishda foydalanib bo'lmaydi. Bunday holda quyidagi teoremadan foydalanish mumkin.

39.2-teorema. $\int_a^{+\infty} |f(x)|dx$ integral yaqinlashsa, $\int_a^{+\infty} f(x)dx$ integral ham yaqinlashadi.

Bunda oxirgi integral **absolyut yaqinlashuvchi** deyiladi.

$\int_a^{+\infty} f(x)dx$ yaqinlashuvchi $\int_a^{+\infty} |f(x)|dx$ integral uzoqlashuvchi bo'lganda $\int_a^{+\infty} f(x)dx$ integral **shartli yaqinlashuvchi** deyiladi.

8- misol. $\int_1^{+\infty} \frac{dx}{x^3(1+e^x)}$ tekshirilsin.

Yechish. Integral ostidagi $\frac{1}{x^3(1+e^x)}$ funksiyani $\frac{1}{x^3}$ funksiya bilan taqqoslaymiz. Barcha $x \geq 1$ uchun $\frac{1}{x^3(1+e^x)} \leq \frac{1}{x^3}$ bo'lib, $\int_1^{+\infty} \frac{dx}{x^3}$ yaqinlashganligi (3- misolga qarang) uchun 39.1. teoremaning *a)* bandiga binoan berilgan integral ham yaqinlashadi.

9- misol. $\int\limits_1^{+\infty} \frac{\sqrt{x}}{1+x}$ tekshirilsin.

Yechish. Integral ostidagi $\frac{\sqrt{x}}{1+x}$ funksiyani $\frac{1}{2\sqrt{x}}$ funksiya bilan taqqoslab barcha $x \geq 1$ uchun

$$\frac{1}{2\sqrt{x}} = \frac{\sqrt{x}}{2x} = \frac{\sqrt{x}}{x+x} \leq \frac{\sqrt{x}}{1+x}$$

ga ega bo'lamiz. $\alpha = \frac{1}{2} < 1$ bo'lgani uchun $\int\limits_1^{+\infty} \frac{dx}{2\sqrt{x}}$ integral uzoqlashadi (3-misolga qarang).

39.1. teoremaning *b)* qismiga ko'ra berilgan integral ham uzoqlashadi.

10-misol. $\int\limits_1^{+\infty} \frac{\sin x}{x^4} dx$ integral tekshirilsin.

Yechish. Integral ostidagi $\frac{\sin x}{x^4}$ funksiya $[1, +\infty)$ da ishorasini saqlamaydi. Shuning uchun $\int\limits_1^{+\infty} \left|\frac{\sin x}{x^4}\right| dx$ integralni qaraymiz. $[1, +\infty)$ da

$$0 \leq \left|\frac{\sin x}{x^4}\right| = \frac{|\sin x|}{x^4} \leq \frac{1}{x^4}$$

bajarilib $\int\limits_1^{+\infty} \frac{dx}{x^4}$ (3-misolga qarang) yaqinlashganligi uchun 39.1-teoremaning *a)* bandiga ko'ra $\int\limits_1^{+\infty} \left|\frac{\sin x}{x^4}\right| dx$ yaqinlashadi.

39.2-teoremaga ko'ra berilgan integral ham yaqinlashadi. U absolyut yaqinlashadi.

11-misol. $\int\limits_0^{\infty} \frac{\sin x}{x} dx$

Dirixle integralining shartli yaqinlashuvchiligi ko‘rsatilsin.

Yechish. Integralni ikkita integrallarning yig‘indisi ko‘rinishda tasvirlaymiz:

$$\int_0^{\infty}\frac{\sin x}{x}dx=\int_0^{\frac{\pi}{2}}\frac{\sin x}{x}dx+\int_{\frac{\pi}{2}}^{\infty}\frac{\sin x}{x}dx.$$

$\lim\limits_{x\to 0}\frac{\sin x}{x}=1$ bo‘lgani uchun birinchi integral xos ma’noda ya’ni u aniq integral sifatida mavjud, chunki $\frac{\sin x}{x}$ funksiya $(0;\frac{\pi}{2}]$ oraliqda uzluksiz bo‘lib $x=0$ nuqta uning yo‘qotilishi mumkin bo‘lgan uzilish nuqtasi.

Ikkinchi integralni bo‘laklab integrallaymiz:

$$\int_{\frac{\pi}{2}}^{\infty}\frac{\sin x}{x}dx=\lim_{A\to\infty}\int_{\frac{\pi}{2}}^{A}\frac{\sin x}{x}dx\left|\begin{matrix}\frac{1}{x}=u, & du=-\frac{1}{x^2}dx\\ \sin x dx=dv, & v=-\cos x\end{matrix}\right|=$$

$$=\lim_{A\to\infty}\left[-\frac{\cos x}{x}\Bigg|_{\frac{\pi}{2}}^{A}-\int_{\frac{\pi}{2}}^{A}\frac{\cos x}{x^2}dx\right]=-\lim_{A\to\infty}\frac{\cos A}{A}-\int_{\frac{\pi}{2}}^{\infty}\frac{\cos x}{x^2}dx=-\int_{\frac{\pi}{2}}^{\infty}\frac{\cos x}{x^2}dx.$$

Ixtiyoriy x uchun $\left|\frac{\cos x}{x^2}\right|\le\frac{1}{x^2}$ bajarilib $\int_{\frac{\pi}{2}}^{\infty}\frac{dx}{x^2}$ $(p=2>1)$ integral yaqinlashganligi sababli 39.1-teoremaning *a)* bandiga ko‘ra $\int_{\frac{\pi}{2}}^{\infty}\frac{|\cos x|}{x^2}dx$ integral ham yaqinlashadi, demak $\int_{\frac{\pi}{2}}^{\infty}\frac{\cos x}{x^2}dx$ absolyut yaqinlashadi.

Demak $\int\limits_{\frac{\pi}{2}}^{\infty}\frac{\sin x}{x}dx$ xosmas integral ham absolyut yaqinlashadi.

Bu integral uchun yuritilgan mulohazalarni takrorlab $\int\limits_{\frac{\pi}{2}}^{\infty}\frac{\cos x}{x}dx$ integralni ham yaqinlashishini ko'rsatish mumkin.

Shunday qilib $\int\limits_{0}^{\frac{\pi}{2}}\frac{\sin x}{x}dx$ aniq integral bilan yaqinlashuvchi $\int\limits_{\frac{\pi}{2}}^{\infty}\frac{\sin x}{x}dx$ xosmas integralning yig'indisidan iborat $\int\limits_{0}^{\infty}\frac{\sin x}{x}dx$ integral ham yaqinlashadi.

Endi $\int\limits_{\frac{\pi}{2}}^{\infty}\frac{|\sin x|}{x}dx$ integralning uzoqlashuvchiligini ko'rsatamiz.

Barcha $x\geq\frac{\pi}{2}$ uchun

$$\frac{|\sin x|}{x}\geq\frac{\sin^2 x}{x}=\frac{1-\cos 2x}{2x}$$

bajarilib

$$\int\limits_{\frac{\pi}{2}}^{\infty}\frac{1-\cos 2x}{2x}dx=\frac{1}{2}\lim_{A\to\infty}\int\limits_{\frac{\pi}{2}}^{A}\frac{dx}{x}-\frac{1}{2}\int\limits_{\frac{\pi}{2}}^{\infty}\frac{\cos 2x}{x}dx=\frac{1}{2}\lim_{A\to\infty}\ln x\Big|_{\frac{\pi}{2}}^{A}-\frac{1}{2}\int\limits_{\frac{\pi}{2}}^{\infty}\frac{\cos 2x}{x}dx=$$

$$=\frac{1}{2}\lim_{A\to\infty}\ln A-\frac{1}{2}\ln\frac{\pi}{2}-\int\limits_{\frac{\pi}{2}}^{\infty}\frac{\cos 2x}{2x}dx=\infty$$

xosmas integral uzoqlashganligi uchun 39.1-teoremaning *b)* bandiga ko‘ra $\int_{\frac{\pi}{2}}^{\infty}\frac{|\sin x|}{x}dx$ xosmas integral ham uzoqlashadi, bu yerda $\int_{\frac{\pi}{2}}^{\infty}\frac{\cos 2x}{2x}dx$ integralning yaqilashuvchanligi hisobga olinadi.

$$\int_0^{\infty}\frac{|\sin x|}{x}dx=\int_0^{\frac{\pi}{2}}\frac{|\sin x|}{x}dx+\int_{\frac{\pi}{2}}^{\infty}\frac{|\sin x|}{x}dx$$

yig‘indining birinchi integrali aniq integral bo‘lib ikkinchi integrali uzoqlashuvchi xosmas integal bo‘lganligi sababli $\int_0^{\infty}\frac{|\sin x|}{x}dx$ integral ham uzoqlashadi.

Demak, $\int_0^{\infty}\frac{\sin x}{x}dx$ xosmas integral shartli yaqinlashuvchi.

12-misol. $\int_0^{\infty}\sin x^2dx$, $\int_0^{\infty}\cos x^2dx$ xosmas integralning yaqinlashuvchiligi isbotlansin.

Yechish. $\int_0^{\infty}\sin x^2dx$ integralning yaqinlashuvchiligini ko‘rsatamiz.

$x=\sqrt{t}$, $dx=\frac{dt}{2\sqrt{t}}$ almashtirish olib berilgan integralni ikkita integrallarning yig‘indisi ko‘rinishda tasvirlaymiz:

$$\int_0^{\infty}\sin x^2dx=\frac{1}{2}\int_0^{\infty}\frac{\sin t}{\sqrt{t}}dt=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\sin t}{\sqrt{t}}dt+\frac{1}{2}\int_{\frac{\pi}{2}}^{\infty}\frac{\sin t}{\sqrt{t}}dt,$$

$\lim\limits_{t\to 0}\frac{\sin t}{\sqrt{t}}=\lim\limits_{t\to 0}\frac{\sin t}{t}\cdot\sqrt{t}=1\cdot 0=0$ bo'lganligi sababli xuddi 11-misoldagi kabi birinchi integral aniq integral sifatida mavjud. Ikkinchi integralni bo'laklab integrallaymiz:

$$\int\limits_{\frac{\pi}{2}}^{\infty}\frac{\sin t}{\sqrt{t}}dt\left|\begin{array}{l}\frac{1}{\sqrt{t}}=u,\ \sin t dt=dv\\ du=(t^{-\frac{1}{2}})'dt=-\frac{dt}{2t^{\frac{3}{2}}},v=-\cos t\end{array}\right|=-\frac{\cos t}{\sqrt{t}}\Bigg|_{\frac{\pi}{2}}^{\infty}-\frac{1}{2}\int\limits_{\frac{\pi}{2}}^{\infty}\frac{\cos t}{t^{\frac{3}{2}}}dt=-\frac{1}{2}\int\limits_{\frac{\pi}{2}}^{\infty}\frac{\cos t}{t^{\frac{3}{2}}}dt.$$

Barcha $t\geq\frac{\pi}{2}$ uchun $\left|\frac{\cos t}{t^{\frac{3}{2}}}\right|\leq\frac{1}{t^{\frac{3}{2}}}$ bo'lib $\int\limits_{\frac{\pi}{2}}^{\infty}\frac{dt}{t^{\frac{3}{2}}}$ integral yaqinlashganligi sababli $\int\limits_{\frac{\pi}{2}}^{\infty}\frac{\cos t}{t^{\frac{3}{2}}}dt$ integral absolyut yaqinlashadi.

Shunday qilib $\int\limits_{0}^{\infty}\sin x^2dx$ xosmas integral aniq integral bilan yaqinlashuvchi xosmas integralning yig'indisidan iborat bo'lganligi uchun u yaqinlashadi.

$\int\limits_{0}^{\infty}\cos x^2dx$ xosmas integralning yaqinlashuvchanligi ham shunga o'xshash isbotlanadi.

$\int\limits_{0}^{\infty}\sin x^2dx$, $\int\limits_{0}^{\infty}\cos x^2dx$ integrallar Freneli integrallari deb ataladi va yorug'likning difraksiyasi nazariyasida uchraydi.

Ikkinchi tur xosmas integral uchun ham 39.1 va 39.2 teoremaga o'xshash teoremalar mavjud.

39.1′-teorema. $f(x)$ va $\varphi(x)$ funksiyalar $(a,b]$ oraliqda uzluksiz bo'lib $x=a$ nuqta ularning ikkinchi tur uzilish nuqtasi (maxsus nuqtasi) bo'lsin. Agar $(a,b]$ oraliqning barcha nuqtalarida

$$0\leq f(x)\leq \varphi(x)$$

tengsizlik bajarilsa, u holda: *a)* $\int_a^b \varphi(x)dx$ xosmas integral yaqinlashsa $\int_a^b f(x)dx$ xosmas integral ham yaqinlashadi.

b) $\int_a^b f(x)dx$ xosmas integral uzoqlashsa $\int_a^b \varphi(x)dx$ xosmas integral ham uzoqlashadi.

39.2′-teorema. a nuqta $(a,b]$ oraliqda uzluksiz $f(x)$ funksiyaning maxsus nuqtasi bo'lib $\int_a^b |f(x)|dx$ xosmas integral yaqinlashsa $\int_a^b f(x)dx$ xosmas integral ham yaqinlashadi.

13-misol. $\int_0^1 \frac{dx}{\sqrt[3]{x}+3x^3}$ xosmas integral tekshirilsin. $\int_a^b f(x)dx$ yaqinlashuvchi bo'lib $\int_a^b |f(x)|dx$ uzoqlashsa $\int_a^b f(x)dx$ shartli yaqinlashadi deyiladi.

Yechish. Integral ostidagi $\frac{1}{\sqrt[3]{x}+3x^2}$ funksiya $(0,1]$ oraliqda uzluksiz bo'lib u $x=0$ nuqtada ikkinchi tur uzilishga ega. $(0,1]$ oraliqdagi barcha x lar uchun

$$\frac{1}{\sqrt[3]{x}+3x^2}<\frac{1}{\sqrt[3]{x}}$$

tengsizlik bajarilib $\int_0^1 \frac{dx}{\sqrt[3]{x}}$ xosmas integral yaqinlashgani uchun (4-misolga qarang) 39.1′-teoremaning *a)* bandiga binoan qaralayotgan xosmas integral ham yaqinlashadi.

14-misol. $\int_1^2 \frac{2+\sin x}{(x-1)^2}dx$ xosmas integral tekshirilsin.

Yechish. Integral ostidagi $\frac{2+\sin x}{(x-1)^2}$ funksiya *(1,2]* oraliqda uzluksiz bo'lib u $x=1$ nuqtada ikkinchi tur uzilishga ega va uning surati x ning istalgan qiymatida $2+sinx\geq1$, chunki $sinx\geq-1$.

Shuning uchun. $\frac{2+\sin x}{(x-1)^2}\geq\frac{1}{(x-1)^2}$. Biroq $\int_1^2 \frac{dx}{(x-1)^2}$ uzoqlashadi, chunki $\alpha=2>1$ (4-misol). Demak 39.1′ teoremaning *b)* bandiga ko'ra berilgan integral ham uzoqlashadi.

15-misol. $\int_0^1 \frac{\cos x dx}{\sqrt{x}}$ xosmas integral tekshirilsin.

Yechish. Integral ostidagi $\frac{\cos x}{\sqrt{x}}$ funksiya $x=0$ maxsus nuqtaga ega. $cosx$ funksiya $[0,\pi]$ kesmada ishorasini saqlamaydi, ya'ni u $[0,\frac{\pi}{2})$ da musbat, $(\frac{\pi}{2},\pi]$ da manfiy. Shuning uchun

$$\int_0^\pi \left|\frac{\cos x}{\sqrt{x}}\right|dx$$

Integralni qaraymiz. Barcha x lar uchun $|cosx|\leq1$ ekanini hisobga olsak $[0,\pi]$ oraliqdagi barcha x lar uchun

$$\left|\frac{\cos x}{\sqrt{x}}\right|\leq\frac{1}{\sqrt{x}}$$

ekanligi kelib chiqadi.

$\int_0^1 \frac{dx}{\sqrt{x}}$ xosmas integral yaqinlashganligi (4-misol) uchun 39.1′-teoremaning *a)* bandiga binoan

$$\int_0^\pi \left|\frac{\cos x}{\sqrt{x}}\right|dx$$

xosmas integral ham yaqinlashadi.

39.2′-teoremaga ko‘ra berilgan xosmas integral ham yaqinlashadi. Demak u absolyut yaqinlashadi.

16-misol. $J=\int_0^{\frac{\pi}{2}}\ln\sin x dx$ integralning yaqinlashuvchanligi ko‘rsatilsin va u hisoblansin.

Yechish. $\lim\limits_{x\to 0}\ln\sin x=\infty$ bo‘lgani uchun integral ostidagi funksiya $x=0$ nuqtada ikkinchi tur uzilisha ega, ya’ni funksiya $[0,\frac{\pi}{2}]$ kesmada chegaralanmagan. Shuning uchun berilgan integral ikkinchi tur xosmas integral.

Integralni bo‘laklab integrallaymiz.

$$\int_0^{\frac{\pi}{2}}\ln\sin x dx\left|\begin{array}{l}\ln\sin x=u,\ dx=dv\\ du=\dfrac{\cos x}{\sin x},\ v=x\end{array}\right|=x\ln\sin x\Big|_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}x\cdot\frac{\cos x}{\sin x}dx=$$

$$=\frac{\pi}{2}\ln\sin\frac{\pi}{2}-\lim_{x\to+0}x\ln\sin x-\int_0^{\frac{\pi}{2}}\frac{x}{tgx}dx.$$

$$\ln\sin\frac{\pi}{2}=\ln 1=0,\ \lim_{x\to+0}x\ln\sin x=\lim_{x\to+0}\frac{\ln\sin x}{\frac{1}{x}}=\lim_{x\to+0}\frac{(\ln\sin x)'}{(\frac{1}{x})'}=$$

$$=\lim_{x\to+0}\frac{ctgx}{-\frac{1}{x^2}}=-\lim_{x\to+0}x^2ctgx=-\lim_{x\to+0}\frac{x^2}{tgx}=-\lim_{x\to-0}\frac{x}{tgx}\cdot x=1\cdot 0=0$$

ekanini hisobga olsak

$$\int_0^{\frac{\pi}{2}}\ln\sin x dx=-\int_0^{\frac{\pi}{2}}\frac{x}{tgx}dx$$

tenglikka ega bo‘lamiz.

$\lim\limits_{x\to 0}\frac{x}{\text{tg}x}=1,\ \lim\limits_{x\to\frac{\pi}{2}-0}\frac{x}{\text{tg}x}=\lim\limits_{x\to\frac{\pi}{2}-0}x\text{ctg}x=0$ bo‘lgani uchun oxirgi integral xosmas integral emas, balki oddiy aniq integraldir. Shuning uchun berilgan integral yaqinlashadi.

Endi uni hisoblashga kirishamiz. $x=2t$ almashtirish olamiz. U holda $dx=2dt$, $x=0$ da $t_1=0$ da $x=\frac{\pi}{2}$ da $t_2=\frac{\pi}{4}$ bo‘lib

$$\int_0^{\frac{\pi}{2}}\ln\sin xdx=2\int_0^{\frac{\pi}{4}}\ln\sin 2tdt=2\int_0^{\frac{\pi}{4}}\ln(2\sin t\cos t)dt=2\int_0^{\frac{\pi}{4}}(\ln 2+\ln\sin t+$$

$$+\ln\cos t)dt=2\ln 2\cdot t\Big|_0^{\frac{\pi}{4}}+2\int_0^{\frac{\pi}{4}}\ln\sin tdt+2\int_0^{\frac{\pi}{4}}\ln\cos tdt=$$

$$\frac{\pi}{2}\ln 2+2\int_0^{\frac{\pi}{4}}\ln\sin tdt+2\int_0^{\frac{\pi}{4}}\ln\cos tdt$$

bo‘ladi.

Oxirgi integralda $t=\frac{\pi}{2}-z$ almashtirish olamiz. U holda $dt=-dz$, $t=0$ da $z_1=\frac{\pi}{2}$, $t=\frac{\pi}{4}$ da $z_2=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$ bo‘lib

$$2\int_0^{\frac{\pi}{4}}\ln\cos tdt=-2\int_{\frac{\pi}{2}}^{\frac{\pi}{4}}\ln\cos(\frac{\pi}{2}-z)dz=2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\ln\sin zdz$$

kelib chiqadi.

Shunday qilib,

$$J=\int_{0}^{\frac{\pi}{2}}\ln\sin x dx=\frac{\pi}{2}\ln 2+2\int_{0}^{\frac{\pi}{4}}\ln\sin t dt+2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\ln\sin z dz=$$

$$=\frac{\pi}{2}\ln 2+2\int_{0}^{\frac{\pi}{2}}\ln\sin t dt=\frac{\pi}{2}\ln 2+2J.$$

Bundan $J=\int_{0}^{\frac{\pi}{2}}\ln(\sin x)dx=-\frac{\pi}{2}\ln 2.$

O‘z-o‘zini tekshirish uchun savollar

1. Xosmas integral tushunchasining kelib chiqishiga sabab nima?
2. Xosmas integral qanaqa turlarga bo‘linadi.
3. Birinchi tur xosmas integralni ta‘riflang.
4. Yaqinlashuvchi va uzoqlashuvchi 1-tur xosmas integrallarga misollar keltiring.
5. Birinchi tur xosmas integral uchun taqqoslash teorimalarini ayting.
6. Birinchi tur xosmas integral uchun absolyut va shartli yaqinlashishni ta‘riflang.
7. Funksiyaning maxsus nuqtasi deb nimaga aytiladi?
8. Uzluksiz funksiyaning maxsus nuqtasi nima?
9. Ikkinchi tur xosmas integralni ta‘riflang.
10. Yaqinlashuvchi va uzoqlashuvchi ikkinchi tur xosmas integrallarga misollar keltiring.
11. Ikkinchi tur xosmas integrallar uchun taqqoslash teoremalarini bayon eting.
12. Ikkinchi tur xosmas integral uchun absolyut va shartli yaqinlashini ta‘riflang.

Mustaqil yechish uchun mashqlar

Quyidagi xosmas integrallarni hisoblang.

1. $\int_{1}^{+\infty}\frac{dx}{x^5}$. Javob: $\frac{1}{4}$.

2. $\int_{0}^{+\infty}\frac{dx}{4+x^2}$ Javob: $\frac{\pi}{4}$.

3. $\int\limits_{\infty}^{+\infty} \frac{dx}{x^2+2x+2}$ Javob: π.

4. $\int\limits_{0}^{1} \frac{dx}{\sqrt[3]{x}}$ Javob: $\frac{3}{2}$.

5. $\int\limits_{0}^{2} \frac{dx}{\sqrt{4-x^2}}$ Javob: $\frac{\pi}{2}$.

6. $\int\limits_{4}^{8} \frac{dx}{\sqrt[3]{x-5}}$ Javob: $\frac{3}{2}(\sqrt[3]{9}-1)$.

Quyidagi integrallarning yaqinlashishi yoki uzoqlashishini tekshiring.

7. $\int\limits_{4}^{+\infty} \frac{x^2dx}{x^3+1}$. Javob: uzoqlashadi.

8. $\int\limits_{1}^{+\infty} \frac{xdx}{\sqrt[3]{(1+x^2)^2}}$ Javob: uzoqlashadi.

9. $\int\limits_{2}^{+\infty} \frac{dx}{x^2+\sqrt{x^4+3}}$ Javob: yaqinlashadi.

10. $\int\limits_{0}^{\frac{\pi}{2}} tgxdx$. Javob: uzoqlashadi.

11. $\int\limits_{0}^{5} \frac{dx}{(x-2)^2}$. Javob: uzoqlashadi.

12. $\int\limits_{0}^{5} \frac{dx}{x^2-6x+8}$. Javob: uzoqlashadi.

13. $\int\limits_{2}^{6} \frac{2+\cos x}{(x-2)^3}dx$. Javob: uzoqlashadi.

40. ANIQ INTEGRALLARNI TAQRIBIY HISOBLASH

40. 1. Masalaning qo‘yilishi

$\int_a^b f(x)dx$ aniq integralni hisoblash talab etilsin, bunda *f(x)* *[a;b]* kesmada uzluksiz funksiya. Agar integral ostidagi *f(x)* funksiyaning *F(x)* boshlang‘ich funksiyasini topish imkoni bo‘lsa, u holda berilgan integral

$$\int_a^b f(x)dx = F(b) - F(a)$$

Nyuton-Leybnis formulasi yordamida hisoblanadi.

Biroq hatto uzluksiz funksiyaning ham boshlang‘ich funksiyasi har doim elementar funksiya bo‘lavermasligini ko‘rdik. Masalan,

$$\frac{\sin x}{x}, \sin x^2, \frac{1}{\ln x}, e^{-x^2}$$

ko‘rinishdagi va boshqa ko‘pgina uzluksiz funksiyalarning boshlang‘ich funksiyalari mavjud bo‘lsada, ular elementar funksiya orqali ifodalanmaydi. Bunday hollarda aniq integralning aniq qiymatini Nyuton-Leybnis formulasidan foydalanib topishning iloji bo‘lmasligi mumkin. Aniq integralning aniq qiymatini topishning iloji bo‘lmaganda uni istalgan aniqlikda taqribiy hisoblash imkonini beruvchi usullar mavjud. Aniq integralni taqribiy hisoblash usuli ko‘p hollarda shu integralning geometrik ma‘nosiga ya‘ni $\int_a^b f(x)dx$ aniq integral son qiymat jihatdan yuqoridan uzluksiz $y = f(x)$ $(f(x) \geq 0)$ egri chiziq, quyidan $0x$ o‘qning *[a;b]* kesmasi va yon tomonlardan $x=a$, $x=b$ vertikal to‘g‘ri chiziqlar bilan chegaralangan egri chiziqli trapetsiyaning yuziga tengligiga asoslanadi. Shunga ko‘ra aniq integralni taqribiy hisoblash masalasi egri chiziqli trapetsiyaning yuzini taqribiy hisoblashga keladi.

Aniq integralni taqribiy hisoblash g'oyasi shundan iboratki bunda $y=f(x)$ egri chiziq o'ziga «yaqin» yangi egri chiziqqa almashtiriladi. Natijada izlanayotgan yuza taqriban yangi egri chiziq bilan chegaralangan egri chiziqli trapetsiyaning yuziga teng bo'ladi.

Yangi egri chiziq sifatida odatda egri chiziqli trapetsiyaning yuzi osonlikcha hisoblanadigan egri chiziq tanlanadi. Shu egri chiziqqa bog'liq ravishda u yoki bu taqribiy integrallash formulalariga ega bo'lamiz.

40.2. To'g'ri to'rtburchaklar formulasi

Faraz qilaylik $y=f(x)$ funksiya $[a;b]$ kesmada uzluksiz bo'lib

$$\int_a^b f(x)dx$$

aniq integralni hisoblash talab etilsin.

$[a;b]$ kesmani $a=x_o, x_1, x_2, \ldots x_к, \ldots, x_n=b$ nuqtalar bilan n ta teng qismga ajratamiz.

Har bir bo'lakning uzunligi

$$\Delta x = \frac{b-a}{n}$$

bo'lishi ravshan.

$f(x)$ funksiyaning $x_o, x_1, x_2, \ldots, x_к, \ldots, x_n$ nuqtalardagi qiymatlari $y_o=f(x_o)$, $y_1=f(x_1)$, $y_2=f(x_2), \ldots, y_к=f(x_к), \ldots, y_n=f(x_n)$ larni hisoblaymiz.

$$y_o\Delta x+y_1\Delta x+y_2\Delta x+\ldots+y_{n-1}\Delta x=\sum_{к=0}^{n-1} y_k\Delta x \text{ va}$$

$$y_1\Delta x+y_2\Delta x+y_3\Delta x+\ldots+y_n\Delta x=\sum_{к=1}^{n} y_k\Delta x$$

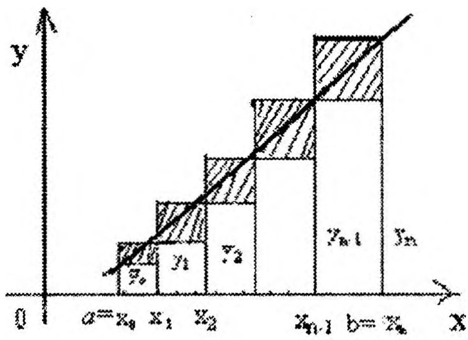
yig'indilarni tuzamiz.

Bu yig'indilarning har biri $[a;b]$ kesmada uzluksiz $f(x)$ funksiyaning integral yig'indisi bo'ladi va shuning uchun ular taqriban integralni ifodalaydilar:

$$\int_a^b f(x)dx \approx \frac{b-a}{n}(y_0 + y_1 + ... + y_{n-1}) = \frac{b-a}{n}\sum_{k=0}^{n-1} y_k \quad (40.1)$$

$$\int_a^b f(x)dx \approx \frac{b-a}{n}(y_1 + y_2 + ... + y_n) = \frac{b-a}{n}\sum_{k=1}^{n} y_k \quad (40.2)$$

Aniq integralni taqribiy hisoblash uchun chiqarilgan bu formulalar **to'g'ri to'rtburchak** formulasi deb yuritiladi.



149-chizma.

149-chizmadan ko'rinib turibdiki, agar *f(x)* funksiya musbat va *[a;b]* kesmada o'suvchi bo'lsa, (40.1) formula berilgan integralning taqribiy qiymatini kami bilan, (40.2) formula esa ortig'i bilan ifodalaydi.

Shu chizmada bu ikki (40.2) va (40.1) yig'indi orasidagi ayirma shtrixlangan n ta to'g'ri to'rtburchak yuzlarining yig'indisiga teng ekanligi ko'rsatilgan:

$$\delta_n = \frac{b-a}{n}[f(b) - f(a)].$$

Binobarin, *[a;b]* kesmada o'suvchi bo'lgan *f(x)* funksiyadan olingan integralni hisoblashda yo'l qo'yilgan xato bu δ_n ayirmadan katta bo'lmaydi. Bo'linishlar soni n ni orttira borib, bu ayirmani istalgancha kichik qilib olish, demak, $\int_a^b f(x)dx$ integralni oldindan berilgan istalgan aniqlik bilan hisoblash mumkin.

Agar $f(x)\geq 0$ funksiya $[a;b]$ kesmada kamaysa $\delta_n = \frac{b-a}{n}[f(a)-f(b)]$ bo'ladi.

Agar $f(x)$ funksiya $[a;b]$ da monoton bo'lmasa, $f'(x)$ hosila esa mavjud bo'lsa va bu kesmada chegaralangan bo'lsa (40.1) va (40.2) formulalarning δ_n xatoligi uchun ushbu baho o'rinlidir:

$$|\delta_n| \leq \frac{M_1(b-a)^2}{2n},$$

bu yerda $M_1 - f'(x)$ ning $[a;b]$ kesmadagi moduli maksimumi.

1-misol. $\int_1^2 \sqrt[3]{x}dx$ integral integrallash oralig'ini $n=8$ bo'lakka bo'lib to'g'ri to'rtburchaklar formulalari yordamida taqribiy hisoblansin. Integral aniq hisoblanib natijaning **absolyut** hamda **nisbiy** xatolari topilsin.

Yechish. Quyidagiga egamiz: $y=\sqrt[3]{x}, n=8, \Delta x = \frac{2-1}{8} = 0{,}125$.

Ushbu jadvalni tuzamiz:

i	0	1	2	3	4	5	6	7	8
x_i	1	1,125	1,250	1,375	1,500	1,625	1,750	1,875	2
y_i	1	1,042	1,073	1,122	1,145	1,117	1,205	1,233	1,260

(40.1) formulaga ko'ra

$$\int_1^2 \sqrt[3]{x}dx \simeq 0{,}125(y_0+y_1+\ldots+y_7)=0{,}125\cdot 8{,}987=1{,}123375,$$

(40.2) formulaga binoan $\int_1^2 \sqrt[3]{x}dx \simeq 0{,}125(y_1+y_2+\ldots,+y_8)=0{,}125\cdot$ $\cdot 9{,}247=1{,}155875$ taqribiy tengliklarga ega bo'lamiz.

Endi berilgan integralni aniq qiymatini Nyuton-Leybnis formulasidan foydalanib topamiz:

$$\int_1^2 \sqrt[3]{x}dx = \int_1^2 x^{\frac{1}{3}}dx = \frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1}\Bigg|_1^2 = \frac{3}{4}\sqrt[3]{x^4}\Bigg|_1^2 = \frac{3}{4}(\sqrt[3]{2^4}-1) \simeq 1,14.$$

Absolyut va nisbiy xatolar mos ravishda (40.1) formula uchun

$|1,14-1,123375|=0,016625 \simeq 0,0167$ va $\frac{0,0167}{1,14}\cdot 100\% = 1,5\%$,

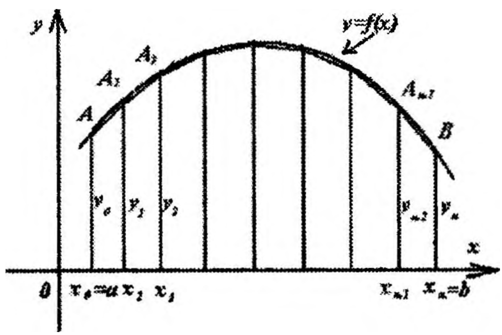
(40.2) formula uchun $|1,14-1,155875|=0,015875 \simeq 0,0158$ va $\frac{0,0158}{1,14}\cdot 100\% \simeq 1,4\%$ bo'ladi.

40.3. Trapetsiyalar formulasi

[a;b] kesmada uzluksiz *f(x)* funksiya berilgan bo'lib $\int_a^b f(x)dx$ aniq integralni hisoblash talab etilsin.

[a;b] kesmani $a= x_o, x_1, x_2, ..., x_\kappa, ..., x_n=b$ nuqtalar bilan uzunligi $\Delta x = \frac{b-a}{n}$ bo'lgan *n* ta teng bo'laklarga ajratamiz.

So'ngra $y=f(x)$ funksiyaning $x_o, x_1, x_2, ..., x_\kappa, ..., x_n$ nuqtalardagi qiymatlari $y_o=f(x_o), y_1=f(x_1), y_2=f(x_2), ..., y_\kappa=f(x_\kappa), ..., y_n=f(x_n)$ larni hisoblaymiz.



150-chizma.

$\int_a^b f(x)dx$ aniq integral *aABb* egri chiziqli trapetsiya yuzini ifodalar edi. *y=f(x)* egri chiziqni unga ichki chizilgan $A_0, A_1, A_2, \ldots, A_{n-1}B$ siniq chiziq bilan almashtiramiz. Bu holda *aABb* egri chiziqli trapetsiyaning yuzi yuqoridan

AA_1, $A_1A_2, \ldots A_{n-1}B$ vatarlar bilan chegaralangan oddiy trapetsiyalar yuzlarining yig'indisiga teng bo'ladi.

Trapetsiyaning yuzi asoslari yig'indisining yarmi bilan balandligi ko'paytmasiga teng bo'lganligi sababli 1-trapetsiyaning yuzi $\frac{y_0+y_1}{2}\Delta x$,

2-sining yuzi $\frac{y_1+y_2}{2}\Delta x$ va hokazo oxirgisining yuzi , $\frac{y_{n-1}+y_n}{2}\Delta x$ ga teng bo'ladi. Shuning uchun

$$\int_a^b f(x)dx \approx \left(\frac{y_0+y_1}{2}\Delta x + \frac{y_1+y_2}{2}\Delta x + \ldots + \frac{y_{n-1}+y_n}{2}\Delta x\right)$$

yoki

$$\int_a^b f(x)dx \approx \frac{b-a}{n}\left(\frac{y_0+y_n}{2} + y_1 + y_2 + \ldots + y_{n-1}\right). \qquad (40.3)$$

(40.3) **trapetsiyalar** formulasi deb ataladi. *n* son qancha katta bo'lsa (40.3) taqribiy tenglikning o'ng tomonidagi yig'indi shuncha katta aniqlik bilan berilgan integralning qiymatini beradi.

(40.3) formulaning xatosi $|\delta_n| \le \frac{M_2(b-a)^3}{12n^2}$ dan oshmaydi, bu yerda M_2- $f''(x)$ funksiyaning *[a;b]* kesmadagi moduli maxsimumi.

2-misol. $\int_0^{1,6} \sin(x^2)dx$ aniq integral *[0; 1,6]* kesmani *n=8* ta teng bo'lakka bo'lib trapetsiyalar formulasi yordamida taqribiy hisoblansin.

Yechish. Integral ostidagi funksiyaning $n=8$ va $\Delta x=\frac{b-a}{n}=\frac{1,6-0}{8}=0,2$ bo'lgandagi qiymatlari jadvalini tuzamiz.

i	x_i	x_i^2	$y_i=\sin(x_i^2)$
0	0	0	0,0000
1	0,2	0,04	0,0400
2	0,4	0,16	0,1593
3	0,6	0,36	0,3523
4	0,8	0,64	0,5972
5	1,0	1	0,8415
6	1,2	1,44	0,9915
7	1,4	1,96	0,9249
8	1,6	2,56	0,5487

$n=8$ bo'lganda (40.3) formulaga binoan

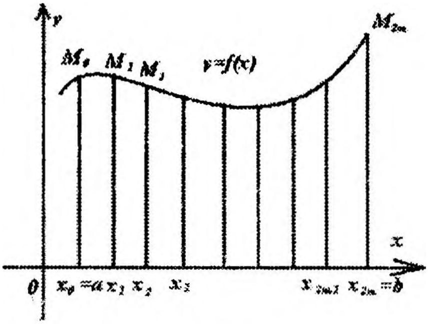
$$\int_0^{1,6}\sin(x^2)dx\approx\Delta x(\frac{y_0+y_8}{2}+y_1+y_2+y_3+y_4+y_5+y_6+y_7)=0,2(\frac{0+0,5487}{2}+$$

$$+0,0400+0,1593+0,3523+0,5972+0,8415+0,9915+0,9249)=0,2\cdot4,1807=0,8362$$

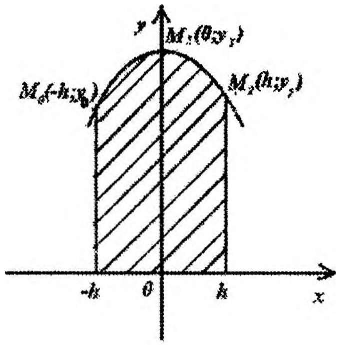
ga ega bo'lamiz.

40.4. Simpson formulasi

$\int_a^b f(x)dx$ integralni hisoblash talab etilsin, bunda $f(x)$ $[a;b]$ kesmada uzluksiz funksiya. $[a;b]$ kesmani juft sondagi $n=2m$ ta teng bo'laklarga bo'lamiz. Birinchi ikkita $[x_o, x_1]$, $[x_1, x_2]$ oraliqlarga mos keluvchi egri chiziqli trapetsiyaning yuzini simmetriya o'qi $0y$ o'qqa parallel bo'lib $M_0(x_o, y_o)$, $M_1(x_1, y_1)$, $M_2(x_2, y_2)$ nuqtalardan o'tuvchi parabola bilan chegaralangan egri chiziqli trapetsiya yuzi bilan almashtiramiz. Yuqorida aytilgan parabolaning umumiy tenglamasi $y=Ax^2+Bx+C$ ko'rinishda bo'ladi. A,B,C o'zgarmas sonlar parabolaning berilgan uchta nuqtadan o'tish shartidan aniqlanadi. Qolgan har juft kesmalar uchun ham shunga o‘xshash parabolalar yasaymiz. Yuqoridan parabolalar bilan chegaralangan parabolik trapetsiyalar yuzlarining yig‘indisi *aABb* egri chiziqli trapetsiya yuzini ya‘ni aniq integral qiymatini taqriban ifodalaydi (151-chizma).



151-chizma.



152-chizma.

Lemma. Agar egri chiziqli trapetsiya $y=Ax^2+Bx+C$ parabola, $0x$ o‘q hamda orasidagi masofa $2h$ ga teng bo‘lgan y_o,y_2 ordinatalar bilan chegaralangan bo‘lsa uning yuzi

$$s=\frac{h}{3}(y_0+4y_1+y_2) \tag{40.4}$$

formula orqali topiladi, bunda y_1-kesmaning o‘rtasiga mos keluvchi egri chiziq ordinatasi.

Isboti. Yordamchi koordinatalar sistemasini 152-chizmada ko‘rsatilganidek qilib joylashtiramiz.

Shartga binoan $y=Ax^2+Bx+C$ parabola M_0, M_1, M_2 nuqtalardan o‘tganliklari sababli bu nuqtalarning koordinatalari parabola tenglamasini qanoatlantiradi:

$$\left.\begin{array}{lll} x_0=-h & \text{bo'lsa} & y_0=Ah^2-Bh+C, \\ x_1=0 & \text{bo'lsa} & y_1=\ C, \\ x_2=h & \text{bo'lsa} & y_2=Ah^2+Bh+C. \end{array}\right\} \tag{40.5}$$

Qaralayotgan parabolik trapetsiyaning yuzini aniq integralning geometrik ma’nosiga asoslanib aniqlaymiz.

$$S=\int_{-h}^{h}(Ax^2+Bx+C)dx=\left[\frac{Ax^3}{3}+\frac{Bx^2}{2}+Cx\right]\Big|_{-h}^{h}=\left(\frac{Ah^3}{3}+\frac{Bh^2}{2}+Ch\right)-$$

$$-\frac{A(-h)^3}{3}-\frac{B(-h)^2}{2}-C(-h)=\frac{2}{3}Ah^3+2Ch=\frac{h}{3}(2Ah^2+6C).$$

Ikkinchi tomondan (40.5) sistemaning ikkinchi tenglamasini *4* ga ko'paytirib barcha tenglamalarni qo'shsak $y_o+4y_1+y_2=2Ah^2+6C$ bo'ladi.

Demak. $S=\frac{h}{3}(y_o+4y_1+y_2)$. Lemma isbot bo'ldi. (40.4) formuladan foydalanib quyidagi taqribiy tengliklarga ega bo'lamiz $(h=\Delta x)$.

$$\int_{a=x_0}^{x_2}f(x)dx\approx\frac{\Delta x}{3}(y_0+4y_1+y_2),$$

$$\int_{x_2}^{x_4}f(x)dx\approx\frac{\Delta x}{3}(y_2+4y_3+y_4),$$

....

$$\int_{x_{2m-2}}^{x_{2m}}f(x)dx\approx\frac{\Delta x}{3}(y_{2m-2}+4y_{2m-1}+y_{2m}).$$

Bu tengliklarni chap va o'ng tomonlarni mos ravishda qo'shsak

$$\int_a^b f(x)dx\approx\frac{\Delta x}{3}(y_o+4y_1+y_2+y_2+4y_3+y_4+\ldots+y_{2m-2}+4y_{2m-1}+y_{2m})$$

yoki

$$\int_a^b f(x)dx\approx\frac{b-a}{6m}[y_o+y_{2m}+2(y_2+y_4+\ldots+y_{2m-2})+4(y_1+y_3+\ldots+y_{2m-1})] \quad (40.6)$$

hosil bo'ladi. Bu formula **Simpson** yoki **parabolalar** formulasi deb ataladi.

Aniq integralni Simpson formulasi yordamida taqribiy hisoblaganda xatolik $|\delta_n|$

$$\frac{M_4(b-a)^5}{180(2m)^4}$$

dan oshmaydi, bu yerda $M_4 - f^{IV}(x)$ funksiyaning *[a;b]* kesmadagi moduli maksimumi.

3-misol. $\int_0^{1,6} \sin(x^2)dx$ integral integrallash oralig'ini *2m=8* bo'lakka bo'lib Simpson formulasi yordamida taqribiy hisoblansin.

Yechish. *2m=8,* $\Delta x = \frac{b-a}{2m} = \frac{1,6-0}{8} = 0,2$ bo'lganda (40.6) formulaga asosan

$$\int_0^{1,6} \sin x^2 dx \approx \frac{1,6-0}{24} [y_0+y_8+2(y_2+y_4+y_6)+4(y_1+y_3+y_5+y_7)]$$

taqribiy tenlikka ega bo'lamiz.

2-misolda *sin(x²)* funksiyaning qiymatlari uchun tuzilgan jadvaldan foydalansak

$$\int_0^{1,6} \sin(x^2)dx \approx \frac{1,6-0}{24} [0+0,5487+2(0,1592+0,5972+0,9915)+$$

$$+4(0,0400+0,3523+0,8415+0,9249)=0,8455.$$

kelib chiqadi.

Shunday qilib $\int_0^{1,6} \sin(x^2)dx$ integralni integrallash oralig'ini *8* ga bo'lib trapetsiyalar formulasi yordamitda taqribiy hisoblanganda uning qiymati 0,8362 ga, Simpson formulasi yordamida hisoblanganda 0,8455 ga tengligini ko'rdik.

Shu integralning 0,00001 aniqlikda hisoblangan jadval qiymati 0,84528 ga tengdir.

Demak, Simpson formulasi trapetsiyalar formulasiga nisbatan qaralayotgan integralning aniqroq qiymatini berar ekan.

4-misol. $\int\limits_{0,5}^{1,5}\frac{e^{0,1x}}{x}dx$ integral Simpson formulasi yordamida $10^{-4}=0,0001$ aniqlikda taqribiy hisoblansin.

Yechish. $\frac{e^x}{x}$ funksiyaning boshlang'ich funksiyasi elementar funksiya bo'lmasligi aytilgan edi. Shuning uchun berilgan integralni bevosita Nyuton-Leybnis formulasidan foydalanib hisoblashning iloji yo'q. Uni taqtibiy hisoblaymiz. Ma'lumki aniq integralni Simpson formulasidan foydalanib taqribiy hisoblaganda $|\delta_n|$ xato

$$\frac{M_4(b-a)^5}{180(2n)^4}$$

dan oshmaydi, bunda $M_4-f^{IV}(x)$ funksiyaning [a;b] kesmadagi moduli maksimumi.

$f(x)=\frac{e^{0,1x}}{x}$ funksiyaning to'rtinchi hosilasi $f^{IV}(x)$ ni topamiz. $f(x)$ ni ketma-ket to'rt marta differensiallab

$$f^{IV}(x)=\frac{P(x)}{x^5}e^{0,1x}$$

ga ega bo'lamiz, bunda

$$P(x)=0,0001x^4-0,004x^3+0,12x^2-2,4x+24.$$

$\varphi(x)=e^{0,1x}$ funksiya [*0,5; 1,5*] kesmada o'suvchi bo'lgani uchun u o'zining eng katta qiymatini *x=1,5* da qabul qiladi: $\varphi(1,5)=e^{0,15}<1,2$.

$\frac{P(x)}{x^5}$ ning absolyut qiymati har bir qo'shiluvchining absolyut qiymatlari yig'indisidan oshmaydi, bunda har bir qo'shiluvchi o'zining eng katta qiymatiga *x=0,5* da erishadi.

$$\left|\frac{P(x)}{x^5}\right| = \left|\frac{0,0001}{x} + \frac{0,004}{x^2} + \frac{0,12}{x^3} - \frac{2,4}{x^4} + \frac{24}{x^5}\right| \le \frac{0,0001}{x} + \frac{0,004}{x^2} + \frac{0,12}{x^3} - \frac{2,4}{x^4} + \frac{24}{x^5} \le$$

$$\le \frac{0,0001}{0,5} + \frac{0,004}{0,5^2} + \frac{0,12}{0,5^3} - \frac{2,4}{0,5^4} + \frac{24}{0,5^5} = 0,0002 + 0,16 + 0,96 + 38,4 + 768 < 808.$$

Shunday qilib, $\left|f^{IV}(x)\right| < 1,2 \cdot 808 < 1000$.

Demak M_4 sifatida *1000* ni olish mumkin.

Berilgan integralni 10^{-4} aniqlikda hisoblash talab etiladi. Bu aniqlikni ta'minlash uchun tegishli amallarni bajarish va yaxlitlashlar oqibatida yo'l qo'yilgan xatolar yig'indisi 10^{-4} dan oshmasligi kerak.

Shu maqsadda $2n$ sonni

$$\left|\delta_n\right| < \frac{1}{2} \cdot 10^{-4} = 5 \cdot 10^{-5}$$

tengsizlik bajariladigan qilib tanlaymiz:

$$\left|\delta_n\right| \le \frac{1^5 1000}{180 \cdot (2n)^4} < 5 \cdot 10^{-5};\ (2n)^4 > \frac{1000 \cdot 10^5}{5 \cdot 180} = \frac{10}{9} \cdot 10^5,$$

$$2n > 10\sqrt[4]{\frac{100}{9}} = 10\sqrt{\frac{10}{3}}.$$

Demak, $2n > 19$ deb olish mumkin.

$2n = 20$ deb olamiz; u holda integrallash odimi

$$h = \frac{b-a}{2n} = \frac{1,5-0,5}{20} = 0,05$$

bo'ladi. Shunday qilib $2n=20$ bo'lganda ya'ni [0,5;1,5] oraliqni teng 20 ta qismga ajratganda

$$\left|\delta_n\right| < 5 \cdot 10^{-5}$$

bo'lar ekan. Yanada aniqroq hisoblashlar natijasida

$$|\delta_n| < 4{,}5 \cdot 10^{-5}$$

ekanini ko'rsatish mumkin.

Agar biz y_i $(i = \overline{0{,}20})$ ni hisoblashda verguldan so'nggi bitta raqamni olsak, ya'ni y_i ni hisoblashda xatolik 10^{-5} dan oshmasa, u holda yaxlitlashlar natijasida yo'l qo'yilgan xato ham 10^{-5} dan oshmaydi.

Shunday qilib umumiy xato ham $4{,}5 \cdot 10^{-5} < 10^{-4}$ dan oshmaydi.

Endi x ning *0,5* dan 1,5 gacha *h=0,05* odimli qiymatlari uchun $y = \dfrac{e^{0{,}1x}}{x}$ funksiyaning qiymatlari jadvalini tuzamiz:

i	x_i	$0{,}1x_i$	$e^{0{,}1x_i}$	y_i	y_i		
					i=0 va i=20	*i-toq bo'lganda*	*i-juft bo'lganda*
0	0,50	0,050	1,05127	2,10254	2,10254		
1	0,55	0,055	1,05654	1,92098		1,92098	
2	0,60	0,060	1,06184	1,76973			1.76973
3	0,65	0,065	1,06716	1,64178		1,64178	
4	0,70	0,070	1,07251	1,53216			1.53216
5	0,75	0,075	1,07788	1,43717		1,43717	
6	0,80	0,080	1,08329	1,35411			1.3541
7	0,85	0,085	1,08872	1,28085		1,28085	
8	0,90	0,090	1,09417	1,21574			1.21574
9	0,95	0,095	1,09966	1,15754		1,15754	
10	1,00	0,100	1,10517	1,10517			1.10517
11	1,05	0,105	1,11071	1,05782		1,05782	
12	1,10	0,110	1,11628	1,01480			1.01480
13	1,15	0,115	1,12187	0,97554		0,97554	
14	1,20	0,120	1,12750	0,93958			0.93958
15	1,25	0,125	1,13315	0,90652		0,90652	
16	1,30	0,130	1,13883	0,87602			0.87602
17	1,35	0,135	1,14454	0,84781		0,84781	
18	1,40	0,140	1,15027	0,82162			0.82162
19	1,45	0,145	1,15604	0,79727		0,79727	
20	1,50	0,150	1,16183	0,77455	0,77455		
jami					2,87709	12,02328	10,62893

(40.6) Simpson formulasiga binoan

$$\int_{0,5}^{1,5}\frac{e^{0,1x}}{x}dx \approx \frac{1}{60}[y_0 + y_{20} + 2(y_2 + y_4 + \ldots + y_{18}) + 4(y_1 + y_3 + \ldots + y_{19})] =$$

$$= \frac{1}{60}(2,87709 + 4\cdot 12,02328 + 2\cdot 10,62893) = \frac{1}{60}\cdot 72,22807 = 1,2038.$$

O'z-o'zini tekshirish uchun savollar

1. Nima uchun aniq integral taqribiy hisoblanadi?
2. Aniq integralni taqribiy hisoblash g'oyasi nimaga asoslangan?
3. To'g'ri to'rtburchaklar formulasini yozing.
4. Trapetsiyalar formulasini yozing.
5. Simpson formulasini yozing.
6. Keltirilgan taqribiy hisoblash formulalaridan qaysi biri aniqroq qiymatni beradi?
7. Xatoni kamaytirish uchun nima qilish kerak?

Mustaqil yechish uchun mashqlar

1. $\int_0^1 \frac{dx}{1+x^2}$ intgralni 0,06 dan katta bo'lmagan xatolik bilan to'g'ri to'rtburchaklar formulasidan foydalanib hisoblang.

Javob: 0,80998(ortig'i bilan), 0,75998 (kami bilan).

2. $\int_0^1 \frac{dx}{1+x^2}$ intgralni trapetsiyalar formulasi bo'yicha $n=10$ deb hisoblang. Javob: 0,78498.

3. $\int_0^1 \frac{dx}{1+x^2}$ integralni $n=2m=4$ deb Simpson formulasi bo'yicha taqribiy hisoblang. Javob: 0,7854.

41. ANIQ INTEGRAL YORDAMIDA YUZNI HISOBLASH

41.1. Dekart koordinatalar sistemasida yuzlarni hisoblash

Agar uzluksiz $f(x)$ funksiya $[a;b]$ kesmada nomanfiy bo'lsa, u holda $y=f(x)$ egri chiziq, ox o'q, $x=a$ va $x=b$ vertikal to'g'ri chiziqlar bilan chegaralangan egri chiziqli trapetsiyaning yuzi

$$Q=\int_a^b f(x)dx \qquad (41.1)$$

ga teng bo'lishini ko'rgan edik (aniq integralning geometrik ma'nosi) (153-chizma).

Agar $[a;b]$ kesmada $f(x)\leq 0$ bo'lsa, u holda aniq integralning 4-xossasiga binoan

$\int_a^b f(x)dx\leq 0$ bo'ladi. Bu holda tegishli egri chiziqli trapetsiyaning yuzi

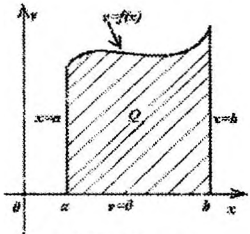
$$-\int_a^b f(x)dx=\int_a^b |f(x)|dx$$

ga teng bo'ladi (154-chizma).

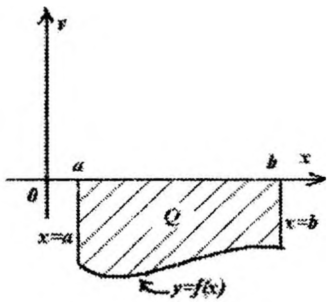
Shuning uchun $f(x)$ funksiya $[a;b]$ kesmada ishorasini o'zgartganda tegishli egri chiziqli trapetsiyaning yuzi

$$Q=\int_a^b |f(x)|dx \qquad (41.2)$$

ga teng bo'ladi.



153-chizma.

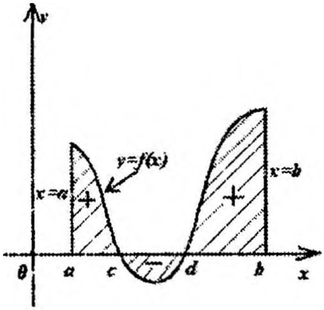


154-chizma.

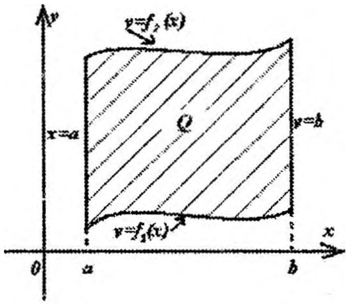
Masalan, 155-chizmada tasvirlangan yuzni quyidagicha topish mumkin:

$$Q=\int_a^c f(x)dx-\int_c^d f(x)dx+\int_d^b f(x)dx.$$

Bu holda (41.1) formuladan foydalanilsa $0x$ o'qning yuqorisida joylashgan figuraning yuzi bilan uning quyisida joylashgan figuraning yuzini ayirmasi topiladi.



155-chizma.



156-chizma.

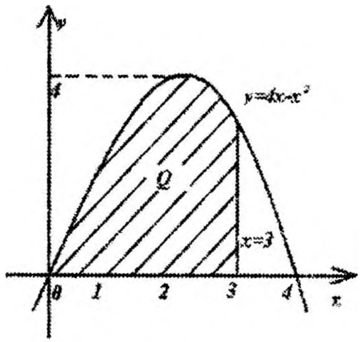
Bizga yuzlarni ayirmasini emas balki ularni yig'indisini topish so'ralganligi sababli $0x$ o'qning pastida joylashgan figuraning yuzini topish uchun integral oldida minus ishora olinadi.

Yuqoridan uzluksiz $y=f_2(x)$, quyidan uzluksiz $y=f_1(x)$, egri chiziqlar bilan va yon tomonlardan $x=a$, $x=b$ $(a<b)$ vertikal to'g'ri chiziqlar bilan chegaralangan egri chiziqli trapetsiyaning yuzini topish uchun

$$Q=\int_a^b f_2(x)dx-\int_a^b f_1(x)dx \quad \text{yoki} \quad Q=\int_a^b [f_2(x)-f_1(x)]dx \qquad (41.3)$$

formulaga ega bo'lamiz(156-chizma).

1-misol. $y=4x-x^2$, $x=3$, $y=0$ chiziqlar bilan chegaralangan figuraning yuzini hisoblang. (157-chizma).

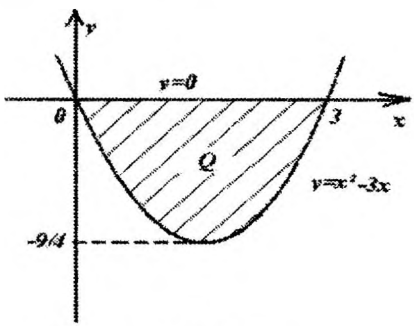


157-chizma.

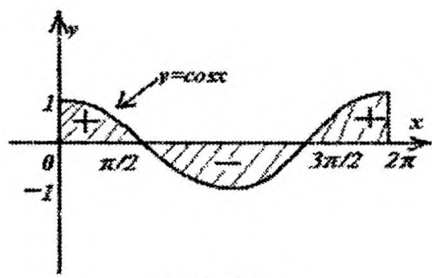
Yechish. (41.1) formuladan foydalanib topamiz.

$$Q=\int_0^3(4x-x^2)dx=\left(4\cdot\frac{x^2}{2}-\frac{x^3}{3}\right)\Big|_0^3=2\cdot 3^2-\frac{3^3}{3}=18-9=18.$$

2-misol. $y=x^2-3x$, $y=0$ chiziqlar bilan chegaralangan figuraning yuzini hisoblang (158-chizma).



158-chizma.



159-chizma.

Yechish. (41.2) formulaga binoan topamiz:

$$Q=-\int_0^3(x^2-3x)dx=-\left(\frac{x^3}{3}-3\cdot\frac{x^2}{2}\right)\Big|_0^3=-\left(\frac{3^3}{3}-3\cdot\frac{3^2}{2}\right)=-(9-13{,}5)=4{,}5.$$

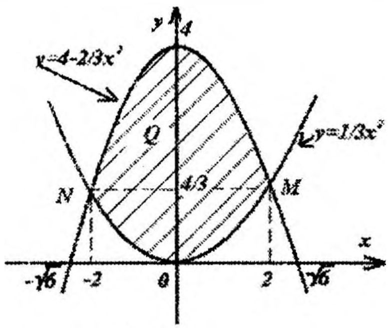
3-misol. $0\le x\le 2\pi$ bo'lganda $y=\cos x$ kosinusoida va $0x$ o'q bilan chegaralangan figuraning yuzi topilsin (159-chizma).

Yechish. $\left[0,\ \frac{\pi}{2}\right]$ da $\cos x\ge 0$, $\left[\frac{\pi}{2},\ \frac{3\pi}{2}\right]$ da $\cos x\le 0$, $\left[\frac{3\pi}{2},\ 2\pi\right]$da $\cos x\ge 0$ ekanligini hisoblaga olib (41.2) formulaga asoslanib topamiz:

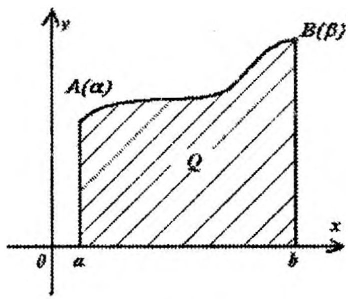
$$Q=\int_0^{2\pi}|\cos x|\,dx=\int_0^{\frac{\pi}{2}}\cos x dx-\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\cos x dx+\int_{\frac{3\pi}{2}}^{2\pi}\cos x dx=\sin x\Big|_0^{\frac{\pi}{2}}-\sin x\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}+\sin x\Big|_{\frac{3\pi}{2}}^{2\pi}=$$

$$=\sin\frac{\pi}{2}-\sin 0-\left(\sin\frac{3\pi}{2}-\sin\frac{\pi}{2}\right)+\sin 2\pi-\sin\frac{3\pi}{2}=1-0-(-1-1)+0-(-1)=4.$$

4-misol. $y=\frac{1}{3}x^2$, $y=4-\frac{2}{3}x^2$ parobalalar bilan chegaralangan figuraning yuzi hisoblansin (160-chizma).



160-chizma.



161-chizma.

Yechish. Integrallash chegaralari a va b ni $y=\frac{1}{3}x^2$ hamda $y=4-\frac{2}{3}x^2$ tenglamalarni birgalikda yechib, ularning kesishish nuqtalari N va M nuqtalarni abssissalarini aniqlash orqali topiladi.

$$\frac{1}{3}x^2=4-\frac{2}{3}x^2,\quad \frac{1}{3}x^2+\frac{2}{3}x^2=4,\qquad x^2=4,\qquad x=\pm 2.$$

Demak, $a=-2$, $b=2$. (41.3) formulaga binoan topamiz:

$$Q=\int_{-2}^{+2}\left[4-\frac{2}{3}x^2-\frac{1}{3}x^2\right]dx=\int_{-2}^{+2}(4-x^2)dx=\left(4x-\frac{x^3}{3}\right)\Big|_{-2}^{2}=$$

$$=4\cdot 2-\frac{2^3}{3}-\left(4\cdot(-2)-\frac{(-2)^3}{3}\right)=8-\frac{8}{3}+8-\frac{8}{3}=\frac{32}{3}.$$

Endi egri chiziq parametrik tenglamalari yordamida berilganda egri chiziqli trapetsiyaning yuzini topish formulasini hosil qilamiz.

Faraz qilaylik egri chiziq $x=\varphi(t)$, $y=\psi(t)$ $\alpha\leq t\leq\beta$ (41.4) parametrik tenglamalari yordamida berilgan bo‘lib $\varphi(t)$, $\psi(t)$ funksiyalar $[\alpha,\beta]$ kesmada uzluksiz va uzluksiz hosilalarga ega bo‘lsin hamda $\varphi(\alpha)=a$, $\varphi(\beta)=b$ bo‘lsin . Agar (41.2) tenglamalar $[a;b]$ kesmada $y=f(x)$ funksiyani aniqlaydi deb faraz qilsak, u holda egri chiziqli trapetsiyaning yuzi (41.1) formulaga binoan

$$Q=\int_a^b f(x)dx=\int_a^b ydx$$

formula bilan hisoblanadi. Oxirgi integralda o‘zgaruvchini almashtiramiz:

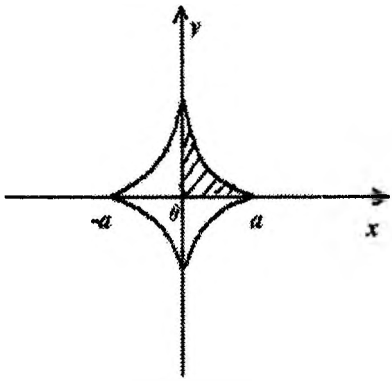
$$x=\varphi(t),\ dx=\varphi'(t)dt,\quad \alpha\leq t\leq\beta, y=f(x)=f[\varphi(t)]=\psi(t).$$

Demak,

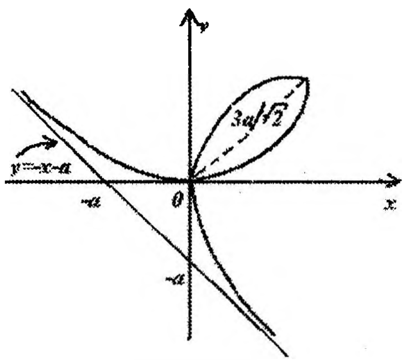
$$Q=\int_\alpha^\beta \psi(t)\varphi'(t)dt. \qquad (41.5)$$

Bu esa parametrik ko‘rinishdagi tenglamalari yordamitda berilgan egri chiziqli trapetsiyaning yuzini hisoblash formulasidir.

5-misol. $x=a\cos^3 t$, $y=a\sin^3 t$ astroida (162-chizma) bilan chegaralangan figuraning yuzini hisoblang.



162-chizma.



163-chizma.

Yechish. t uchun integrallash chegaralarini $x=a\cos^3 t$ tenglamadan topamiz:

$$x=0 \text{ da } \quad a\cos^3 t=0 \quad \cos t=0, \;\; t=\frac{\pi}{2},$$

$$x=a \text{ da } \quad a\cos^3 t=a \quad \cos t=1, \;\; t=0.$$

Astroidani koordinata o'qlariga nisbatan simmetrikligini hisobga olsak 162-chizmadagi shtrixlangan yuz izlanayotgan yuzning to'rtdan birini tashkil etadi. Shuning uchun (41.5) formulaga binoan astroida bilan chegaralangan figura yuzining to'rtdan biri uchun quyidagiga ega bo'lamiz.

$$\frac{1}{4}Q=\int_{\frac{\pi}{2}}^{0} a\sin^3 t(a\cos^3 t)'dt=\int_{\frac{\pi}{2}}^{0} a\sin^3 t3a\cos^2 t(\cos t)'dt=-3a^2\int_{\frac{\pi}{2}}^{0}\sin^4 t\cos^2 tdt=$$

$$=3a^2\int_0^{\frac{\pi}{2}}(\sin^2 t)^2\cos^2 tdt=3a^2\int_0^{\frac{\pi}{2}}\left(\frac{1-\cos 2t}{2}\right)^2\cdot\frac{1+\cos 2t}{2}dt=$$

$$=\frac{3a^2}{8}\int_0^{\frac{\pi}{2}}(1-\cos 2t)(1+\cos 2t)(1-\cos 2t)dt==\frac{3a^2}{8}\int_0^{\frac{\pi}{2}}(1-\cos^2 2t)(1-\cos 2t)dt=$$

$$=\frac{3a^2}{8}\int_0^{\frac{\pi}{2}}\sin^2 2t(1-\cos 2t)dt=\frac{3a^2}{8}\int_0^{\frac{\pi}{2}}\sin^2 2tdt-\frac{3a^2}{8}\int_0^{\frac{\pi}{2}}\sin^2 2t\cos 2tdt=$$

$$=\frac{3a^2}{8}\cdot\int_0^{\frac{\pi}{2}}(1-\cos 4t)dt\frac{1}{2}-\frac{3a^2}{8}\cdot\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin^2 2td(\sin 2t)=\frac{3a^2}{16}(t-\frac{1}{4}\sin 4t)\Big|_0^{\frac{\pi}{2}}-$$

$$-\frac{3a^2}{16}\cdot\frac{\sin^3 2t}{3}\Big|_0^{\frac{\pi}{2}}=\frac{3a^2}{16}\left(\frac{\pi}{2}-\frac{1}{4}\sin 2\pi\right)-\frac{3a^2}{48}(\sin^3\pi-\sin^3 0)=$$

$$=\frac{3a^2}{16}\left(\frac{\pi}{2}-0\right)-\frac{3a^2}{48}(0-0)=\frac{3a^2\pi}{32}.$$

Bundan $\quad Q=4\cdot\frac{3a^2\pi}{32}=\frac{3a^2\pi}{8}.$

6-misol. $x=\frac{3at}{1+t^3};\ y=\frac{3at^2}{1+t^3}$ Dekart yaprog‘i sirtmog‘ining yuzini hisoblang (163-chizma).

Yechish.

$$y\cdot dx=\psi(t)\varphi'(t)dt=\frac{3at^2}{1+t^3}\cdot\left(\frac{3at}{1+t^3}\right)'dt=\frac{9a^2t^2}{1+t^3}\cdot\frac{1+t^3-3t^3}{(1+t^3)^2}dt=\frac{9a^2t^2(1-2t^3)}{(1+t^3)^3}dt.$$

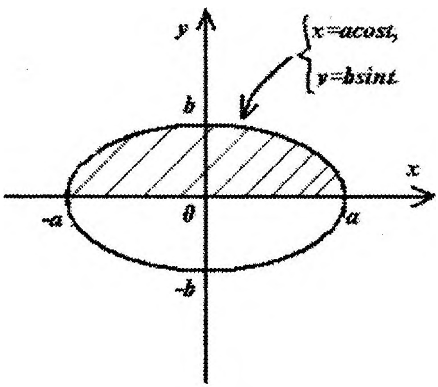
Koordinatalar boshida egri chiziq o‘zini-o‘zi kesadi, ya‘ni koordinatalar boshi egri chiziqning maxsus (qaytish)nuqtasidir. Egri chiziq bu nuqtadan *t=0* da va *t=*∞ bo‘lganda o‘tadi, ya‘ni integrallash chegaralari *0* va ∞ ga teng. Shuning uchun:

$$Q = 9a^2 \int_{-\infty}^{0} \frac{t^2(1-2t^3)dt}{(1+t^3)^3} \left| \begin{matrix} 1+t^3 = z, & 3t^2dt = dz, & t^2dt = \frac{1}{3}dz \\ t=0, \; da, & z=1, t=\infty, & da \quad z=\infty. \end{matrix} \right| =$$

$$= 9a^2 \int_{-\infty}^{1} \frac{(1-2(z-1))dz}{3z^3} = 3a^2 \int_{-\infty}^{1} \frac{3-2z}{z^3} dz = 3a^2 \int_{-\infty}^{1} (3z^{-3} - \frac{2}{z^2})dz =$$

$$= 3a^2 \left(-\frac{3}{2} z^{-2} + \frac{2}{z} \right) \Bigg|_{-\infty}^{1} = 3a^2 \left(-\frac{3}{2} + 2 \right) - 0 = \frac{3a^2}{2}.$$

7-misol. $x = a\cos t$, $y = b\sin t$ ellips bilan chegaralangan figuraning yuzini hisoblang (164-chizma).



164-chizma.

Yechish. Ellips koordinata o'qlariga nisbatan simmetrikligini hisobga olsak 164-chizmadagi shtrixlangan yuz izlanayotgan yuzning yarmini tashkil etadi. Shuning uchun uni hisoblab ikkilantirsak ellips bilan chegaralangan figuraning yuzi hosil bo'ladi. Bu yerda x ning qiymati $-a$ dan a gacha o'zgaradi. U holda t ning qiymatini $x = a\cos t$ dan aniqlasak u π dan 0 gacha o'zgaradi. (41.5) formulaga asosan ellips bilan chegaralangan figura yuzining yarmi uchun quyidagiga ega bo'lamiz:

$$\frac{Q}{2}=\int_{\pi}^{0} b\sin t(a\cos t)' dt=-ab\int_{\pi}^{0}\sin^2 t dt=-ab\int_{\pi}^{0}\frac{1-\cos 2t}{2}=-\frac{ab}{2}\int_{\pi}^{0}(1-\cos 2t)dt=$$

$$=-\frac{ab}{2}\left(t-\frac{\sin 2t}{2}\right)\Big|_{\pi}^{0}=\frac{-ab}{2}\cdot 0+\frac{ab}{2}\left(\pi-\frac{\sin 2\pi}{2}\right)=\frac{ab\pi}{2}.$$

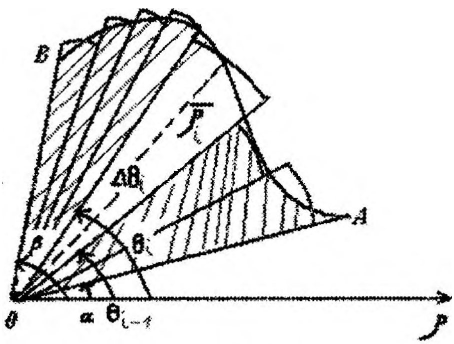
Bundan $Q=ab\pi$ ga ega bo'lamiz.

Xususiy holda $a=b=R$ bo'lganda oxirgi tenglikdan doiraning yuzini topish formulasi $Q=\pi R^2$ ni hosil qilamiz.

41.2. Qutb koordinatalar sistemasida egri chiziqli sektorning yuzi

Qutb koordinatalar sistemasida egiri chiziq $\rho=f(\theta)$ tenglama bilan berilgan bo'lsin, bu yerda $f(\theta)$ funksiya $[\alpha,\beta]$ kesmada uzluksiz.

$\rho=f(\theta)$ egri chiziq, $\theta=\alpha$, $\theta=\beta$ radius-vektorlar bilan chegaralangan *OAB* sektorning yuzini topamiz (165-chizma).



165-chizma.

Berilgan sektorni $\alpha=\theta_0$, $\theta=\theta_1$, $\theta=\theta_2$,... $\theta=\theta_{n-1}$, $\theta_n=\beta$ radius-vektorlar bilan *n* ta bo'lakka ajratamiz. O'tkazilgan radius-vektorlar orasidagi burchaklarni $\Delta\theta_1$, $\Delta\theta_2$,..., $\Delta\theta_n$ bilan belgilaymiz. θ_{i-1} bilan θ_i orasidagi biror $\overline{\theta}_i$ burchakka mos radius-vektorning uzunligini $\overline{\rho}_i$ orqali belgilaymiz. Radiusi $\overline{\rho}_i$ va markaziy burchagi

$\Delta\theta_i$ bo‘lgan doiraviy sektorni qaraymiz. Uni yuzi $\Delta Q_i = \frac{1}{2}\overline{\rho}_i^2 \Delta\theta_i$

kabi topilishi ma‘lum.

Ushbu yig‘indi

$$Q_n = \frac{1}{2}\sum_{i=i}^{n} \overline{\rho}^2 \Delta\theta_i = \frac{1}{2}\sum_{i=i}^{n} [f(\overline{\theta}_i)]^2 \Delta\theta_i.$$

«zinapoyasimon» sektorning yuzini, ya‘ni OAB sektorni yuzining taqribiy qiymatini ifodalaydi.

Bu yig‘indi $[\alpha,\beta]$ kesmada $\rho^2=[f(\theta)]^2$ uzluksiz funksiyaning integral yig‘indisi bo‘lganligi sababli u $\lambda=\max\Delta\theta_i\to 0$ da

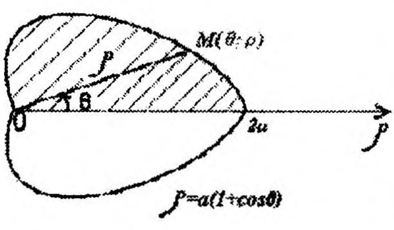
$$\frac{1}{2}\int_{\alpha}^{\beta} \rho^2 d\theta$$

limitga ega. Shunday qilib OAB sektorning yuzi

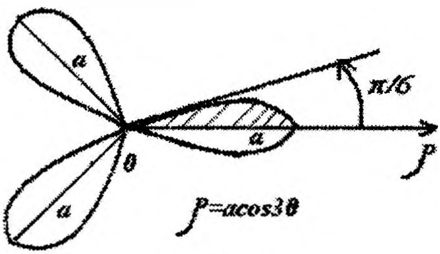
$$Q = \frac{1}{2}\int_{\alpha}^{\beta} \rho^2 d\theta = \frac{1}{2}\int_{\alpha}^{\beta} [f(\theta)]^2 d\theta \quad (41.6)$$

formula yordamida hisoblanar ekan.

8-misol. $\rho=a(1+\cos\theta)$ kardioida bilan chegaralangan figuraning yuzini hisoblang (166-chizma).



166-chizma.



167-chizma.

Yechish. Figura qutb o‘qiga nisbatan simmetrik. Shuning uchun figuraning qutb o‘qi ρ dan yuqorida joylashgan qismining yuzini topsak izlanayotgan yuzning yarmi topiladi. θ o‘zgaruvchi 0 dan π gacha o‘zgarganda ixtiyoriy $M(\theta;\rho)$ nuqta kardioidaning ρ o‘qdan yuqorida joylashgan qismini chizadi. (41.6) formulaga binoan quyidagiga ega bo'lamiz:

$$\frac{Q}{2}=\frac{1}{2}\int_0^{\pi}[a(1+\cos\theta)]^2d\theta=\frac{a^2}{2}\int_0^{\pi}[a(1+2\cos\theta+\cos^2\theta)d\theta=$$

$$=\frac{a^2}{2}(\theta+2\sin\theta)\Big|_0^{\pi}+\frac{a^2}{2}\int_0^{\pi}\frac{1+\cos 2\theta}{2}d\theta=\frac{a^2}{2}(\pi+2\sin\pi)+\frac{a^2}{4}(\theta+\frac{1}{2}\sin 2\theta)\Big|_0^{\pi}=$$

$$=\frac{a^2\pi}{2}+\frac{a^2}{4}(\pi+\frac{1}{2}\sin 2\pi)=\frac{3a^2\pi}{4}.$$

Bundan

$$Q=\frac{3a^2\pi}{2}$$

ga ega bo'lamiz.

9-misol. $\rho=a\cos 3\theta$ egri chiziq bilan chegaralangan figuraning yuzini hisoblang (167-chizma).

Yechish. θ o'zgaruvchi $0,\ \frac{2\pi}{3},\ \frac{4\pi}{3}$ qiymatlarni qabul qilganda ρ o'zining eng katta a qiymatiga erishadi. θ o'zgaruvchi $\frac{\pi}{6},\ \frac{\pi}{2},\ \frac{5\pi}{6}$ qiymatlarni qabul qilganda egri chiziq qutbdan o'tadi $(\rho=0)$.

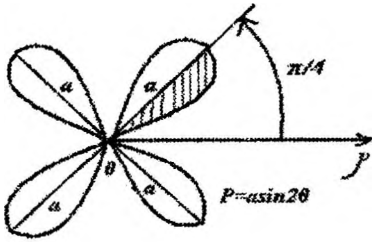
167-chizmadagi shrixlangan yuz uch yaproqli gul bilan chegaralangan yuzning oltidan bir qismini tashkil etadi. Shuning uchun (41.6)ga binoan quyidagiga ega bo'lamiz.

$$\frac{Q}{6}=\frac{1}{2}\int_0^{\frac{\pi}{6}}(a\cos 3\theta)^2d\theta=\frac{a^2}{2}\int_0^{\frac{\pi}{6}}\cos^2 3\theta d\theta=$$

$$=\frac{a^2}{2}\int_0^{\frac{\pi}{6}}\frac{1+\cos 6\theta}{2}d\theta=\frac{a^2}{4}(\theta+\frac{1}{6}\sin 6\theta)\Big|_0^{\frac{\pi}{6}}=\frac{a^2}{4}\left(\frac{\pi}{6}+\frac{1}{6}\sin\pi\right)=\frac{a^2\pi}{24}.$$

Bundan $Q=\frac{a^2\pi}{4}$.

10-misol. $\rho=a\sin 2\theta$ egri chiziq bilan chegaralangan figuraning yuzini hisoblang (168-chizma).



168-chizma.

Yechish. θ o'zgaruvchi $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $\frac{5\pi}{4}$ va $\frac{7\pi}{4}$ qiymatlarni qabul qilganda ρ o'zining eng katta a qiymatiga erishadi. θ o'zgaruvchi 0, $\frac{\pi}{2}$, π, $\frac{3\pi}{2}$, qiymatlarni qabul qilganda $(\rho=0)$ egri chiziq qutbdan o'tadi.

168-chizmadagi shtrixlangan yuz to'rt yaproqli gul bilan chegaralangan yuzning sakkizdan bir qismini tashkil etadi. Shuning uchun (41.6) formulaga binoan quyidagiga ega bo'lamiz:

$$\frac{Q}{8}=\frac{1}{2}\int_0^{\frac{\pi}{4}}(a\sin 2\theta)^2 d\theta=\frac{a^2}{2}\int_0^{\frac{\pi}{4}}\sin^2 2\theta d\theta=$$

$$=\frac{a^2}{2}\int_0^{\frac{\pi}{4}}\frac{1-\cos 4\theta}{2}d\theta=\frac{a^2}{4}(\theta-\frac{1}{2}\sin 4\theta)\Big|_0^{\frac{\pi}{4}}=\frac{a^2}{4}\left(\frac{\pi}{4}-\frac{1}{2}\sin\pi\right)=\frac{a^2}{4}\cdot\frac{\pi}{4}=\frac{\pi a^2}{16}.$$

Bundan $$Q=8\cdot\frac{\pi a^2}{16}=\frac{\pi a^2}{2}.$$

O'z-o'zini tekshirish uchun savollar

1. $y=f(x)\geq 0$, $y=0$, $x=a$, $x=b$ $(a<b)$ chiziqlar bilan chegaralangan figuraning yuzini topish formulasini yozing.
2. $y=f(x)$, $y=0$, $x=a$, $x=b$ chiziqlar bilan chegaralangan figuraning yuzini topish formulasini yozing.
3. $y=f_1(x)$, $y=f_2(x)$, $(f_1(x)\leq f_2(x))$, $x=a$, $x=b$ chiziqlar bilan chegaralangan figuraning yuzini topish formulasini yozing.
4. Egri chiziq parametrik tenglamalari bilan berilganda figuraning yuzini topish formulasini yozing.
5. Qutb koordinatalar sistemasida berilgan egri chiziq bilan chegaralangan egri chiziqli sektorning yuzini hisoblash formulasini yozing.
6. Astroida bilan chegaralangan figuraning yuzini topish formulasini yozing.
7. Dekart yaprog'i sirtmog'ining yuzini topish formulasini yozing.
8. Ellips bilan chegaralangan figura yuzini topish formulasini yozing.
9. Kardioida bilan chegaralangan figura yuzini topish formulasini yozing.
10. $\rho=a\cos 3\theta$ egri chiziq bilan chegaralangan figuraning yuzini topish formulasini keltirib chiqaring.
11. $\rho=a\sin 2\theta$ egri chiziq bilan chegaralangan figuraning yuzini topish formulasini keltirib chiqaring.

Mustaqil yechish uchun mashqlar

Quyidagi chiziqlar bilan chegaralangan figuraning yuzlari topilsin.

1. $y=\ln x$, $x=e$, $y=0$. Javob: 1.
2. $y=e^x$, $y=e^{-x}$, $x=2$. Javob: $(e-e^{-1})^2$.
3. $y=x(x-1)(x-2)$, $y=0$. Javob: $\frac{1}{2}$.
4. $y=x^2+4x$, $y=x+4$. Javob: $20\frac{5}{6}$.

5. $y = \dfrac{a^3}{x^2 + a^2}$, $y = 0$. Javob: πa^2.

6. $x = a(t - \sin t)$, $y = a(1 - \cos t)$ sikloidaning bir arkasi va $0x$ o‘q bilan chegaralangan figuraning yuzini hisoblang. Javob: $3\pi a^2$.

Quyidagi chiziqlar bilan chegaralangan figura yuzini hisoblang.

7. $\rho = a \sin 3\theta$. Javob: $Q = \dfrac{\pi a^2}{4}$.

8. $\rho = a \cos 2\theta$. Javob: $Q = \dfrac{\pi a^2}{2}$.

Egri chiziqning eng katta va eng kichik qo‘shni radius-vektorlari orasidagi figuraning yuzini toping.

9. $\rho = 3 + \cos 2\theta$. Javob: $Q = \dfrac{19\pi}{8}$.

10. $\rho = 2 + \sin 3\theta$. Javob: $Q = \dfrac{3\pi}{4}$.

42. EGRI CHIZIQ YOYINING UZUNLIGI

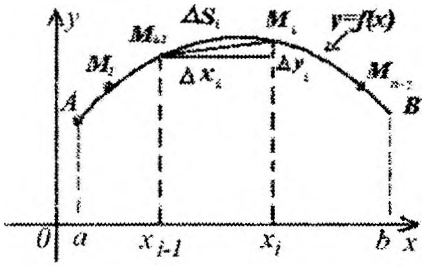
42.1. Dekart koordinatalar sistemasida egri chiziq yoyining uzunligi

Egri chiziq $y=f(x)$ tenglama bilan berilganda uning $x=a$ va $x=b$ $(a<b)$ vertikal to'g'ri chiziqlar orasidagi AB yoyining uzunligini topamiz, bunda $y=f(x)$ funksiya $[a,b]$ kesmada uzluksiz deb faraz qilamiz. AB yoyda abssissalari $a=x_0$, x_1, $x_2,\dots$, x_{i-1}, $x_i,\dots$, $x_n=b$ bo'lgan A, M_1, $M_2,\dots$, M_{i-1}, $M_i,\dots,B$ nuqtalarni olamiz va AM_1, $M_1M_2,\dots$, $M_{i-1}M_i,\dots$, $M_{n-1}B$ vatarlarni o'tkazamiz hamda ularning uzunliklarini mos ravishda ΔS_1, $\Delta S_2,\dots$, ΔS_n bilan belgilaymiz.

Bu holda AB yoyga ichki chizilgan $AM_1M_2\dots$, $M_{n-1}B$ siniq chiziq hosil bo'ladi. Siniq chiziqning uzunligi

$$S_n=\sum_{i=1}^{n}\Delta S_i$$

ga teng bo'ladi (169-chizma).



169-chizma.

AB yoyning uzunligi deb unga ichki chizilgan siniq chiziqning eng katta zvenosi uzunligi nolga intilgandagi

$$S=\lim_{\max\Delta S_i\to 0}\sum_{i=1}^{n}\Delta S_i \qquad (42.1)$$

limitga aytiladi.

Endi $f(x)$ funksiya $[a,b]$ kesmada uzluksiz $f'(x)$ hosilaga ega bo'lganda yoy uzunligini hisoblash formulasini chiqaramiz.

$\Delta y_i = f(x_i) - f(x_{i-1})$, $\Delta x_i = x_i - x_{i-1}$ deb belgilasak.

$$\Delta S_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2} = \sqrt{1 + \left(\frac{\Delta y_i}{\Delta x_i}\right)^2}\,\Delta x_i$$

bo'ladi. Lagranj teoremasiga asosan

$$\frac{\Delta y_i}{\Delta x_i} = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = f'(z_i),$$

Bunda $x_{i-1} < z_i < x_i$. Demak $\Delta S_i = \sqrt{1 + [f'(z_i)]^2}\ \Delta x_i$ va $S_n = \sum_{i=1}^{n} \sqrt{1 + [f'(z_i)]^2}\ \Delta x_i$ bo'ladi.

S_n $[a,b]$ kesmada uzluksiz $\sqrt{1 + [f'(x)]^2}$ funksiyaning integral yig'indisi bo'lganligi sababli $\lambda = \max \Delta x_i \to 0$ da uning limiti mavjud va quyidagi aniq integralga teng.

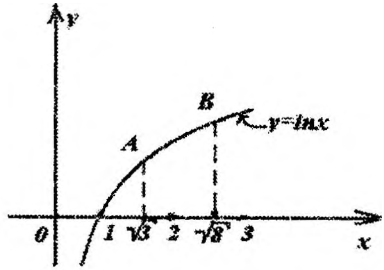
$$S = \lim_{\max \Delta S_i \to 0} \sum_{i=1}^{n} \Delta S_i = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} \sqrt{1 + [f'(z_i)]^2}\,\Delta x_i = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx.$$

Shunday qilib AB yoy uzunligini hisoblash uchun

$$S = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx = \int_a^b \sqrt{1 + y'^2}\,dx \qquad (42.2)$$

formulani hosil qildik.

1-misol. $y = \ln x$ egri chiziq yoyining $x = \sqrt{3}$ dan $x = \sqrt{8}$ gacha bo'lgan qismi uzunligini hisoblang.



170-chizma.

Yechish. $1+y'^2=1+(\ln x)'^2=1+\frac{1}{x^2}=\frac{1+x^2}{x^2}$.

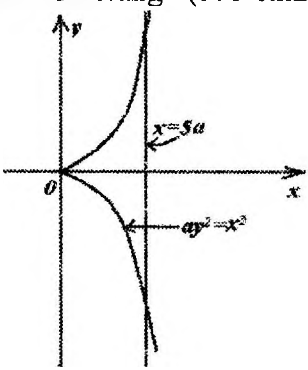
(42.2) formulaga binoan quyidagiga ega bo'lamiz:

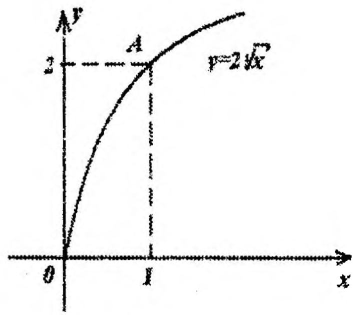
$$S=\int_{\sqrt{3}}^{\sqrt{8}}\frac{\sqrt{1+x^2}}{x}dx\left|\begin{array}{l}1+x^2=t^2, x=\sqrt{t^2-1}, dx=\dfrac{tdt}{\sqrt{t^2-1}}\\ x=\sqrt{3}, da\, t=2; x=\sqrt{8}da, t=3\end{array}\right|=\int_2^3\frac{t^2}{t^2-1}dt=$$

$$=\int_2^3\frac{t^2-1+1}{t^2-1}dt=\int_2^3\left(1+\frac{1}{t^2-1}\right)dt=\left(t+\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|\right)\Bigg|_2^3=3+\frac{1}{2}\ln\frac{2}{4}-2-\frac{1}{2}\ln\frac{1}{3}=$$

$$=1+\frac{1}{2}\left(\ln\frac{1}{2}-\ln\frac{1}{3}\right)=1+\frac{1}{2}\ln\frac{1}{2}\cdot\frac{3}{1}=1+\frac{1}{2}\ln\frac{3}{2}.$$

2-misol. $ay^2=x^3$ $(a>o)$ yarim kubik parabola yoyining koordinatalar boshidan abssissasi $x=5a$ nuqtagacha bo'lgan qismining uzunligini hisoblang (171-chizma).





171-chizma. *172-chizma.*

Yechish. $(ay^2)'=(x^3)'$; $2ay\cdot y'=3x^2$.

$$y'=\frac{3x^2}{2ay};\quad y'^2=\frac{9x^4}{4a^2y^2}=\frac{9x^4}{4a^2\cdot\frac{x^3}{a}}=\frac{9x}{4a}.$$

(42.2) formulaga binoan quyidagiga ega bo‘lamiz:

$$S=\int_0^{5a}\sqrt{1+y'^2}\,dx=\int_0^{5a}\sqrt{1+\frac{9}{4a}x}\,dx=\int_0^{5a}\left(1+\frac{9}{4a}x\right)^{\frac12}dx=\frac{4a}{9}\cdot\frac{(1+\frac{9}{4a}x)^{\frac32}}{\frac32}\Bigg|_0^{5a}=$$

$$=\frac{8a}{27}\left[\left(1+\frac{9}{4a}5a\right)^{\frac32}-1\right]=\frac{8a}{27}\left[\left(\frac{49}{4}\right)^{\frac32}-1\right]=\frac{8a}{27}\left[\left(\left(\frac{7}{2}\right)^2\right)^{\frac32}-1\right]=$$

$$=\frac{8a}{27}\left(\frac{7^3}{2^3}-1\right)=\frac{8a}{27}\cdot\frac{7^3-2^3}{8}=\frac{335}{27}a.$$

3-misol. $y=2\sqrt{x}$ parabola yoyining $x=0$ dan $x=1$ gacha bo‘lgan qismi OA yoyining uzunligini hisoblang (172-chizma).

Yechish.

$$y'=2\cdot\frac{1}{2\sqrt{x}}=\frac{1}{\sqrt{x}},\quad 1+y'^2=1+\frac{1}{x}=\frac{x+1}{x}=\frac{(x+1)^2}{x(x+1)}=\frac{(x+1)^2}{x^2+x}.$$

(42.2) formulaga binoan

$$S=\int_0^1\sqrt{\frac{(x+1)^2}{x^2+x}}dx=\int_0^1\frac{x+1}{\sqrt{x^2+x}}dx=\frac12\int_0^1\frac{2x+1+1}{\sqrt{x^2+x}}dx=\frac12\int_0^1\frac{2x+1}{\sqrt{x^2+x}}dx+\frac12\int_0^1\frac{dx}{\sqrt{x^2+x}}=$$

$$\frac12\int_0^1\frac{(x^2+x)'dx}{\sqrt{x^2+x}}+\frac12\int_0^1\frac{dx}{\sqrt{x^2+x+\frac14-\frac14}}=\int_0^1\frac{d(x^2+x)}{2\sqrt{x^2+x}}+\frac12\int_0^1\frac{dx}{\sqrt{(x+\frac12)^2-\frac14}}=$$

$$=\sqrt{x^2+x}\Big|_0^1+\frac{1}{2}\cdot\ln\left|x+\frac{1}{2}+\sqrt{(x+\frac{1}{2})^2-\frac{1}{4}}\right|\Big|_0^1=\sqrt{2}+\frac{1}{2}\ln\left(\frac{3}{2}+\sqrt{2}\right)-\frac{1}{2}\ln\frac{1}{2}=$$

$$=\sqrt{2}+\frac{1}{2}\cdot\ln(\frac{3}{2}+\sqrt{2})\cdot 2==\sqrt{2}+\frac{1}{2}\ln(3+2\sqrt{2}).$$

4-misol. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ (162-chizma) astroida yoyining uzunligi topilsin.

Yechish. Agar astroidaning $0x$ o‘qqa nisbatan simmetrikligini hisobga olsak uning uzunligini to‘rtdan biri ya’ni birinchi chorakdagi qismining yoy uzunligini topishning o‘zi kifoya.

Birinchi chorakda $y=(a^{\frac{2}{3}}-x^{\frac{2}{3}})^{\frac{3}{2}}$.

$$y'=\frac{3}{2}(a^{\frac{2}{3}}-x^{\frac{2}{3}})^{\frac{1}{2}}\cdot(-\frac{2}{3})x^{-\frac{1}{3}}=-x^{-\frac{1}{3}}(a^{\frac{2}{3}}-x^{\frac{2}{3}})^{\frac{1}{2}}$$

va

$$\sqrt{1+y'^2}=\sqrt{1+x^{-\frac{2}{3}}(a^{\frac{2}{3}}-x^{\frac{2}{3}})}=\sqrt{x^{-\frac{2}{3}}a^{\frac{2}{3}}}=x^{-\frac{1}{3}}a^{\frac{1}{3}}=(\frac{a}{x})^{\frac{1}{3}}.$$

(42.2) formulaga binoan

$$l=4\int_0^a x^{-\frac{1}{3}}a^{\frac{1}{3}}dx=4a^{\frac{1}{3}}\lim_{\varepsilon\to 0}\int_\varepsilon^a x^{-\frac{1}{3}}dx=4a^{\frac{1}{3}}\lim_{\varepsilon\to 0}\frac{x^{\frac{2}{3}}}{\frac{2}{3}}\Bigg|_\varepsilon^a=6a^{\frac{1}{3}}\lim_{\varepsilon\to 0}(a^{\frac{2}{3}}-\varepsilon^{\frac{2}{3}})=6a.$$

Endi egri chiziq tenglamasi $x=\varphi(t)$, $y=\Psi(t)$, $\alpha\leq t\leq\beta$ (42.3) parametrik ko‘rinishda berilgan bo‘lsa shu egri chiziq yoyining uzunligini topamiz. $\varphi(x)$, $\Psi(x)$ funksiyalar $[\alpha,\beta]$ kesmada uzluksiz va uzluksiz hosilalarga ega deb faraz qilamiz. $\varphi(\alpha)=a$, $\varphi(\beta)=b$ bo‘lsin. (42.3) tenglamalar $[a,b]$ kesmada biror $y=f(x)$ funksiyani aniqlaydi deb faraz qilsak izlanayotgan yoy uzunligi

$$S=\int_a^b\sqrt{1+y'^2}\,dx$$

formula yordamida topiladi (42.2 formula). Bu integralda $x=\varphi(t)$,

$dx=\varphi'(t)dt$ almashtirishni bajarib va $y'=\frac{\psi'(t)}{\varphi'(t)}$ ekanini hisobga olib quyidagiga ega bo'lamiz:

$$S=\int_{\alpha}^{\beta}\sqrt{1+\left[\frac{\psi'(t)}{\varphi'(t)}\right]^2}\,\varphi'(t)dt$$

yoki

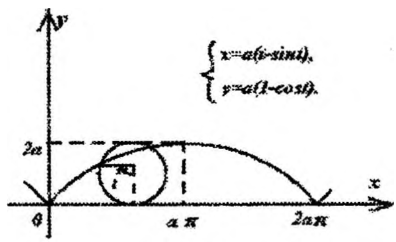
$$S=\int_{\alpha}^{\beta}\sqrt{[\varphi'(t)]^2+[\psi'(t)]^2}\,dt.$$

Shunday qilib (42.3) parametrik tenglamalari yordamida berilgan egri chiziq yoyining uzunligi

$$S=\int_{\alpha}^{\beta}\sqrt{x'^2+y'^2}\,dt=\int_{\alpha}^{\beta}\sqrt{[\varphi'(t)]^2+[\psi'(t)]^2}\,dt \qquad (42.4)$$

formula yordamida topilar ekan.

5-misol. $x=a(t\text{-}\sin t)$, $y=a(1\text{-}\cos t)$ sikloida bitta arkasining uzunligi topilsin (173-chizma).



173-chizma.

Yechish. To'g'ri chiziq bo'ylab harakatlanayotgan g'ildirakning (aylananing) berilgan aniq nuqtasini traektoriyasi sikloida deb ataluvchi egri chiziqni chizadi. G'ildirak to'liq bir marta aylanganda uning aniq nuqtasini chizgan chizig'i sikloidaning bitta arkasini tashkil etadi. G'ildirakni radiusi *a* hamda harakat boshlanganga qadar g'ildirakning aniq nuqtasi koordinatalar boshida bo'lgan deb faraz qilinsa sikloidani ifodalovchi egri chiziq 173-chizmada tasvirlangan.

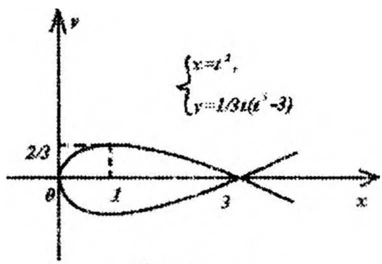
$$x'=a(1-\cos t),\ y'=a\sin t,\ x'^2+y'^2=a^2(1-\cos t)^2+a^2\sin^2 t=$$
$$=a^2(1-2\cos t+\cos^2 t+\sin^2 t)=a^2(2-2\cos t)=2a^2(1-\cos t)=$$
$$=4a^2\sin^2\frac{t}{2}=(2a\sin\frac{t}{2})^2.$$

$x=0$ da $t=0$ va $x=2a\pi$ da $t=2\pi$.

(42.4) formulaga binoan:

$$S=\int_0^{2\pi}\sqrt{x'^2+y'^2}\,dt=\int_0^{2\pi}\sqrt{(2a\sin\frac{t}{2})^2}\,dt=2a\int_0^{2\pi}\sin\frac{t}{2}\,dt=2a\cdot 2(-\cos\frac{t}{2})\Big|_0^{2\pi}=$$
$$=4a(-\cos\pi+\cos o)=8a.$$

6-misol. $x=t^2, y=\frac{1}{3}t(t^2-3)$ egri chiziqning $0x$ o‘q bilan kesishish nuqtalari orasidagi yoyi uzunligini hisoblang (174-chizma).



174-chizma.

Yechish. t ning qanday qiymatlarida egri chiziq Ox o‘qni kesib o‘tishini aniqlaymiz: $y=\frac{1}{3}t(t^2-3)=0$ tenglamadan $t_1=0$, $t_2=\pm\sqrt{3}$ kelib chiqadi. Bularni $x=t^2$ ga qo‘ysak $x_1=0$ va $x_2=3$ egri chiziq bilan Ox o‘qni kesishish nuqtalarining abssissalari hosil bo‘ladi. Yoyning yarmi uchun integrallash chegaralari $\alpha=0$ va $\beta=\sqrt{3}$ bo‘ladi.

$$x'=(t^2)'=2t,$$

$$y' = \left(\frac{1}{3}t(t^2-3)\right)' = \frac{1}{3}(t^3-3t)' = \frac{1}{3}(3t^2-3) = t^2-1.$$

$$\sqrt{x'^2+y'^2} = \sqrt{(2t)^2+(t^2-1)^2} = \sqrt{4t^2+t^4-2t^2+1} = \sqrt{(t^2+1)^2} = t^2+1$$

Demak, (42.4) formulaga binoan

$$\frac{S}{2} = \int_0^{\sqrt{3}}(t^2+1)dt = \left(\frac{t^3}{3}+t\right)\Bigg|_0^{\sqrt{3}} = \frac{(\sqrt{3})^3}{3}+\sqrt{3} = 2\sqrt{3},$$

bundan $S = 4\sqrt{3}$ hosil bo'ladi.

7-misol. $\begin{cases} x = e^t\cos t, \\ y = e^t\sin t \end{cases}$ egri chiziq yoyining $t=0$ dan $t=\ln\pi$ gacha qismining uzunligini hisoblang.

Yechish. $x' = e^t\cos t - e^t\sin t$, $y' = e^t\sin t + e^t\cos t$,

$x'^2+y'^2 = e^{2t}(\cos t-\sin t)^2 + e^{2t}(\sin t+\cos t)^2 = e^{2t}(\cos^2 t - 2\cos t\sin t + \sin^2 t + \sin^2 t + 2\sin t\cos t + \cos^2 t) = e^{2t}(2\cos^2 t + 2\sin^2 t) =$
$2e^{2t}(\cos^2 t+\sin^2 t) = 2e^{2t}$.

(42.4) formulaga binoan quyidagiga ega bo'lamiz:

$$S = \int_0^{\ln\pi}\sqrt{2e^{2t}}dt = \sqrt{2}\int_0^{\ln\pi}e^t dt = \sqrt{2}e^t\Big|_0^{\ln\pi} = \sqrt{2}(e^{\ln\pi}-e^0) = \sqrt{2}(\pi-1).$$

($a^{\log_a b} = b$ ayniyatdan foydalandik).

8-misol. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (164-chizma) ellipsning yoy uzunligi topilsin.

Yechish. Ellipsning $x = a\cos t$, $y = b\sin t$, $0 \le t \le 2\pi$ parametrik tenglamalariga o'tamiz.

t bo'yicha differensiallab

$$x_t' = -a\sin t,\ y_t' = b\cos t$$

ni va bundan

$$\sqrt{x_t'^2 + y_t'^2} = \sqrt{a^2 \sin^2 t + b^2 \cos^2 t} = \sqrt{a^2 - a^2 \cos^2 t + b^2 \cos^2 t} =$$

$$= a\sqrt{1 - \frac{a^2 - b^2}{a^2}\cos^2 t} = a\sqrt{1 - \varepsilon^2 \cos^2 t}$$

ni hosil qilamiz.

Bu yerda $\varepsilon = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}$ -ellipsning ekssentrisiteti.

Shunday qilib (42.4) formulaga binoan

$$l = a\int_0^{2\pi} \sqrt{1 - \varepsilon^2 \cos^2 t}\,dt = 4a\int_0^{\frac{\pi}{2}} \sqrt{1 - \varepsilon^2 \cos^2 t}\,dt.$$

$\int_0^{\frac{\pi}{2}} \sqrt{1 - \varepsilon^2 \cos^2 t}\,dt$ elementar funksiyalar orqali ifodalanmaydi va u ikkinchi tur elliptik integral deb ataladi. $t = \frac{\pi}{2} - z$ almashtirish olib integralni standart ko‘rinishi

$$\int_0^{\frac{\pi}{2}} \sqrt{1 - \varepsilon^2 \cos^2 t}\,dt = \int_0^{\frac{\pi}{2}} \sqrt{1 - \varepsilon^2 \sin^2 t}\,dt = E(\varepsilon)$$

ga ega bo‘lamiz.

Shunday qilib ellisning yoy uzunligini topish uchun

$$l = 4aE(\varepsilon)$$

formulaga ega bo‘lamiz.

Odatda $\varepsilon = \sin\alpha$ deb olib

$$E_1(\alpha) = E_1(\arcsin\varepsilon) = E(\varepsilon)$$

funksiyaning qiymatlari jadvalidan foydalaniladi.

Masalan, Agar $a=10$ va $b=6$ bo‘lsa, u holda

$$\varepsilon = \frac{\sqrt{10^2 - 6^2}}{10} = 0.8 = \sin 53^0.$$

Ikkinchi tur elliptik integralning qiymatlari jadvalidan $l = 4 \cdot 10 \cdot E_1(53^0) = 40 \cdot 1{,}2776 \approx 51{,}1$ ga ega bo'lamiz.

42.2. Qutb koordinatalar sistemasida egri chiziq yoyining uzunligi

Qutb koordinatalar sistemasida egri chiziq $\rho = f(\theta)$ tenglama bilan berilgan bo'lsin, bu yerdagi $f(\theta)$ funksiya $[\alpha, \beta]$ kesmada uzluksiz va uzluksiz hosilaga ega bo'lgan funksiya, ρ-qutb radiusi, θ-qutb burchagi. Shu egri chiziq yoyining uzunligini topish formulasini hosil qilamiz. Qutb koordinatalaridan dekart koordinatalariga o'tish formulasi $x = \rho\cos\theta$, $y = \rho\sin\theta$ ga ρ o'rniga $f(\theta)$ ni qo'ysak $x = f(\theta)\cos\theta$, $y = f(\theta)\sin\theta$ tenglamalar hosil bo'ladi. Bu tenglamalarga egri chiziqning parametrik tenglamalari deb qarab yoy uzunligini hisoblash formulasi (42.4) dan foydalanamiz:

$$x'_\theta = (f(\theta)\cos\theta)' = f'(\theta)\cos\theta - f(\theta)\sin\theta,$$
$$y'_\theta = (f(\theta)\sin\theta)' = f'(\theta)\sin\theta + f(\theta)\cos\theta,$$
$$x'^2_\theta + y'^2_\theta = (f'(\theta)\cos\theta - f(\theta)\sin\theta)^2 +$$
$$+ (f'(\theta)\sin\theta + f(\theta)\cos\theta)^2 = f'^2(\theta)\cos^2\theta -$$
$$-2f'(\theta)f(\theta)\cos\theta\sin\theta + f^2(\theta)\sin^2\theta + f'^2(\theta)\sin^2\theta + 2f'(\theta)f(\theta)\cos\theta\sin\theta +$$
$$+ f^2(\theta)\cos^2\theta = f'^2(\theta)(\cos^2\theta + \sin^2\theta) + f^2(\theta)(\cos^2\theta + \sin^2\theta) =$$
$$= f'^2(\theta) + f^2(\theta) = \rho'^2 + \rho^2.$$

Demak,

$$S = \int_a^B \sqrt{\rho'^2 + \rho^2}\, d\theta. \tag{42.5}$$

Bu qutb koordinatalar sistemasida berilgan egri chiziqning yoy uzunligini topish formulasidir.

9-misol. $\rho = a(1 + \cos\theta)$ kardioidaning uzunligini hisoblang (166-chizma).

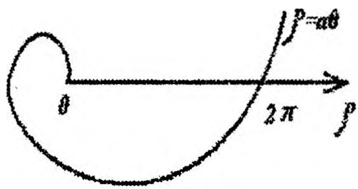
Yechish. Kardioida qutb o'qiga nisbatan simmetrik joylashganligi sababli qutb burchagi θ 0 dan π gacha o'zgarsa (42.5) formula yordamida kardioida uzunligining yarmi topiladi:

$$\frac{S}{2}=\int_0^{\pi}\sqrt{[a(1+\cos\theta)']^2+[a(1+\cos\theta)]^2}d\theta=\int_0^{\pi}\sqrt{a^2(\sin^2\theta+1+2\cos\theta+\cos^2\theta)}d\theta=$$

$$=a\int_0^{\pi}\sqrt{2(1+\cos\theta)}d\theta=a\int_0^{\pi}\sqrt{4\cos^2\frac{\theta}{2}}d\theta=a\int_0^{\pi}2\cos\frac{\theta}{2}d\theta=2a\cdot2\sin\frac{\theta}{2}\Big|_0^{\pi}=$$

$$=4a(\sin\frac{\pi}{2}-\sin 0)=4a; S=2\cdot4a=8a.$$

10-misol. $\rho=a\theta$ Arximed spirali birinchi o'rami yoyining uzunligini hisoblang (175-chizma).



175-chizma.

Yechish. $\rho'=a, \sqrt{\rho'^2+\rho^2}=\sqrt{a^2+a^2\theta^2}=a\sqrt{1+\theta^2}$.

(42.5) formulaga binoan:

$$S=a\int_0^{2\pi}\sqrt{1+\theta^2}d\theta\left|\begin{matrix}\sqrt{1+\theta^2}=u, d\theta=dv\\ du=\dfrac{\theta d\theta}{\sqrt{1+\theta^2}}, v=\theta\end{matrix}\right|=a\theta\sqrt{1+\theta^2}\Big|_0^{2\pi}-a\int_0^{2\pi}\frac{\theta^2 d\theta}{\sqrt{1+\theta^2}}=$$

$$=a2\pi\sqrt{1+4\pi^2}-a\int_0^{2\pi}\frac{1+\theta^2-1}{\sqrt{1+\theta^2}}d\theta=a2\pi\sqrt{1+4\pi^2}-a\int_0^{2\pi}\sqrt{1+\theta^2}d\theta+a\int_0^{2\pi}\frac{d\theta}{\sqrt{1+\theta^2}}=$$

$$=a2\pi\sqrt{1+4\pi^2}-S+a\ln(\theta+\sqrt{1+\theta^2})\Big|_0^{2\pi}=a2\pi\sqrt{1+4\pi^2}-S+a\ln(2\pi+\sqrt{1+4\pi^2}).$$

Bundan

$$2S=2a\pi\sqrt{1+4\pi^2}+a\ln(2\pi+\sqrt{1+4\pi^2})$$

yoki

$$S = a\pi\sqrt{1+4\pi^2} + \frac{a}{2}\ln(2\pi + \sqrt{1+4\pi^2}).$$

11-misol. $\rho = a\cos\theta$ aylana yoyining uzunligini hisoblang.

Yechish. Avvalo berilgan egri chiziqni aylana ekanligiga ishonch hosil qilish uchun tenglamani ρ ga ko'paytiramiz. U holda

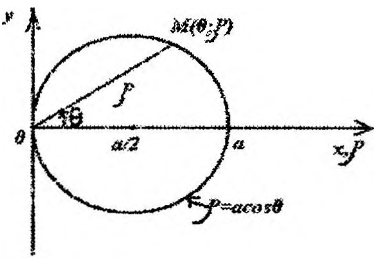
$$\rho^2 = a\rho\cos\theta$$

tenglik hosil bo'ladi. Qutb va dekart koordinatalari orasidagi bog'-lanish

$$\rho^2 = x^2 + y^2, \qquad x = \rho\cos\theta$$

ni hisobga olsak $x^2+y^2=ax$, $x^2-ax+y^2=0$ yoki $(x-\frac{a}{2})^2 + y^2 = (\frac{a}{2})^2$

tenglamaga ega bo'lamiz. Bu tenglama markazi $(\frac{a}{2};\ 0)$ nuqtada bo'lib radiusi $\frac{a}{2}$ ga teng aylana tenglamasidir (176-chizma).



176-chizma.

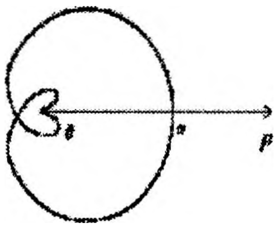
Aylananing qutb o'qidan yo'qorida joylashgan yarmining uzunligini hisoblaymiz:

$$\frac{S}{2} = \int_0^{\frac{\pi}{2}} \sqrt{\rho^2 + \rho'^2}\,d\theta = \int_0^{\frac{\pi}{2}} \sqrt{a^2\cos^2\theta + a^2\sin^2\theta}\,d\theta = a\int_0^{\frac{\pi}{2}} d\theta = a\theta\Big|_0^{\frac{\pi}{2}} = a\frac{\pi}{2}.$$

Bundan $S = a\cdot\frac{\pi}{2}\cdot 2 = a\pi$ o'zimizga ma'lum aylanani uzunligini topish formulasiga ega bo'lamiz.

12-misol. $\rho = a\sin^4(\varphi/4)$ yopiq egri chiziqning uzunligi topilsin.

Yechish. $\rho = a\sin^4(\varphi/4)$ juft funksiya bo'lgani uchun berilgan egri chiziq qutb o'qqa nisbatan simmetrik joylashgan. $\sin^4(\varphi/4)$ funksiya 4π davrga ega davriy funksiya $(\sin^4(\frac{\varphi+4\pi}{4}) = \sin^4(\varphi/4))$ bo'lgani sababli φ qutb burchagi 0 dan 2π gacha o'zgarganda qutb radiusi 0 dan a gacha o'sadi va egri chiziqning yarmini chizadi. Qutb burchagi 2π dan 4π gacha o'zgarganda egri chiziqning chizilgan birinchi yarmiga nisbatan simmetrik bo'lgan ikkinchi yarmi ham chiziladi (177-chizma).



177-chizma.

$$\rho'_\varphi = (a\sin^4(\varphi/4))' = a\cdot 4\sin^3(\varphi/4)\cdot\cos(\varphi/4)\cdot\frac{1}{4} = a\sin^3(\varphi/4)\cos(\varphi/4)$$

va $0 \le \varphi \le 2\pi$ bo'lganda

$$\sqrt{\rho^2+\rho'^2} = \sqrt{a^2\sin^8(\varphi/4)+a^2\sin^6(\varphi/4)\cos^2(\varphi/4)} =$$
$$= a\sqrt{\sin^6(\varphi/4)(\sin^2(\varphi/4)+\cos^2(\varphi/4))} = a\sin^3(\varphi/4).$$

Demak (42.5) formulaga binoan

$$l=2a\int_0^{\frac{\pi}{2}}\sin^3(\varphi/4)d\varphi\left|\begin{matrix}\varphi=4t\\ d\varphi=4dt\end{matrix}\right|=2a\int_0^{\frac{\pi}{2}}\sin^3 t\cdot 4dt=$$

$$=8a\int_0^{\frac{\pi}{2}}\sin^3 tdt=8a\int_0^{\frac{\pi}{2}}(1-\cos^2 t)\sin tdt=$$

$$=-8a\int_0^{\frac{\pi}{2}}(1-\cos^2 t)d(\cos t)=-8a(\cos t-\frac{\cos^3 t}{3})\Bigg|_0^{\frac{\pi}{2}}=8a(1-\frac{1}{3})=\frac{16}{3}a.$$

$\left.\begin{matrix}x=\varphi(t),\\ y=\psi(t),\\ z=h(t)\end{matrix}\right|$ $(\alpha\le t\le\beta)$ parametrik tenglamalari yordamida berylgan fazoviy egri chiziqning yoyi uzunligi

$$S=\int_\alpha^\beta\sqrt{[\varphi'(t)]^2+[\psi'(t)]^2+[h'(t)]^2}\,dt$$

formula yordamida topilishini ta'kidlab o'tamiz.

O'z-o'zini tekshirish uchun savollar

1. Yoy uzunligini ta'riflang.
2. $y=f(x)$ tenglama bilan berilgan egri chiziq yoyining uzunligini topish formulasini yozing.
3. Egri chiziq parametrik tenglamalar bilan berilganda uning yoyini uzunligini topish formulasini yozing.
4. Qutb koordinatalar sistemasida berilgan egri chiziq yoyining uzunligini topish formulasini yozing.
5. Sikloida nima va uning uzunligi qanday topiladi?
6. Kardioidaning uzunligini hisoblash formulasini chiqaring.
7. Arximed spirali birinchi o'rami uzunligini hisoblash formulasini chiqaring.
8. Aylana uzunligini hisoblash formulasini chiqaring.
9. Ellips uchun yoy uzunligini topish formulasini chiqarishga harakat qiling.
10. $y=f(x)$ egri chiziq yoyining uzunligini hisoblash uchun chiqarilgan formulalarda $f(x)$ funksiya qanday shartlarni qanoatlantiradi?

Mustaqil yechish uchun mashqlar

1. $y^2=\frac{1}{9}(x+1)^3$ egri chiziq yoyining $0y$ o'q ajratgan qismining uzunligini hisoblang. Javob: $\frac{2}{3}(5\sqrt{5}-8)$.

2. $y=\ln(1-x^2)$ egri chiziq yoyining $x=0$ dan $x=\frac{1}{3}$ gacha qismining uzunligini hisoblang. Javob: $\ln 2-\frac{1}{3}$.

3. $y^2=4(x-1)$ egri chiziq yoyining $x=1$ dan $x=2$ gacha bo'lgan qismining uzunligini hisoblang. Javob: $\sqrt{3}+\frac{1}{2}\ln\frac{9}{5}$.

4. $\left.\begin{array}{l} x=4\sin t+3\cos t, \\ y=4\cos t-3\sin t \end{array}\right\},0\le t\le\pi$ egri chiziq yoyining uzunligini hisoblang. Javob: 5π.

5. $\begin{cases} x=t^2-1, \\ y=\frac{t^3}{3}-t. \end{cases}$ egri chiziq sirtmog'i uzunligini hisoblang.

Javob: $4\sqrt{3}$.

6. $\rho=a\sin\theta$ egri chiziq yoyi uzunligini hisoblang. Javob: πa.

43. ANIQ INTEGRAL YORDAMIDA JISMNING HAJMI VA SIRTINI TOPISH

43.1. Aylanish jismining hajmi

$y=f(x)$ funksiya $[a;b]$ kesmada uzluksiz va nomanfiy funksiya bo'lsin. $y=f(x)$ egri chiziq, $0x$ o'q va $x=a$, $x=b$ vertikal to'g'ri chiziqlar bilan chegaralangan $aABb$ egri chiziqli trapetsiyani $0x$ o'q atrofida aylanishi natijasida hosil bo'lgan aylanish jismining hajmini topamiz. $[a;b]$ kesmani

$$a=x_0< x_1< x_2<\ldots<x_{i-1}< x_i<\ldots<x_{n-1}<x_n=b$$

nuqtalar yordamida n ta ixtiyoriy

$$[x_0,x_1],[x_1,x_2],\ldots,[x_{i-1},x_i],\ldots,[x_{n-1},x_n]$$

kesmalarga ajratamiz. Har bir $[x_{i-1}, x_i]$ $(i=\overline{1,n})$ kesmachada ixtiyoriy z_i nuqtani olib funksiyaning bu nuqtadagi qiymati $f(z_i)$ ni hisoblaymiz.

Keyin asosi $\Delta x_i=x_i-x_{i-1}$ bo'lib balandligi $f(z_i)$ bo'lgan $PMNQ$ to'g'ri to'rtburchak yasaymiz.

Bu to'g'ri to'rtburchak $0x$ o'q atrofida aylanganda asosining radiusi $f(z_i)$ bo'lib balandligi Δx_i bo'lgan doiraviy silindr hosil bo'ladi. Bu silindrning hajmi $\vartheta_i=\pi R^2H=\pi f^2(z_i)\ \Delta x_i$ formula yordamida topilishi ravshan. Barcha (n ta) silindrlar hajmlarining yig'indisi qaralayotgan aylanish jismi hajmi V_x ning taqribiy qiymatini beradi, ya'ni $V_x\approx\sum_{i=i}^{n}\pi f^2(z_i)\Delta x_i$

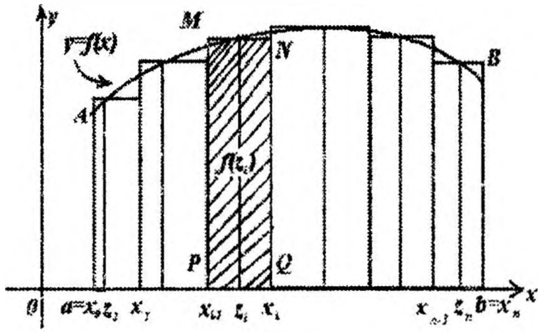
Ikkinchi tomondan bu yig'indi $[a;b]$ kesmada uzluksiz $\pi f^2(x)$ funksiya uchun integral yig'indi bo'lganligi sababli u $\lambda=\max\Delta x_i\to 0$ da $\pi\int_a^b f^2(x)dx$ aniq integralga teng limitga ega.

Shunday qilib aylanish jismining hajmi

$$V_x = \pi \int_a^b f^2(x)dx \qquad (43.1)$$

formula yordamida topilar ekan.

Endi *aABb* egri chiziqli trapetsiyani *Oy* o'q atrofida aylanishi natijasida hosil bo'lgan jismning hajmini topamiz (178-chizma). *PMNQ* to'g'ri to'rtburchak *Oy* o'q atrofida aylanishi natijasida hosil bo'lgan silindrning hajmi asosining radiusi x_i, balandligi $f(z_i)$ bo'lgan silindrning hajmidan asosining radiusi x_{i-1}, balandligi $f(z_i)$ bo'lgan silindr hajmining ayrilganiga teng, ya'ni



178-chizma.

$$\overline{V_i} = \pi x_i^2 f(z_i) - \pi x_{i-1}^2 f(z_i) = \pi f(z_i)(x_i^2 - x_{i-1}^2) =$$
$$= \pi f(z_i)(x_i + x_{i-1})(x_i - x_{i-1}) = \pi f(z_i)(x_i + x_{i-1})\Delta x_i .$$

$\pi \sum_{i=i}^{n} f(z_i)(x_i + x_{i-1})\Delta x_i$. yig'indi *aABb* egri chiziqli trapetsiyani *Oy* o'q atrofida aylanishi natijasida hosil bo'lgan jismning hajmi V_y ni taqribiy qiymatini beradi.

Demak $V_y \approx \pi \sum_{i=i}^{n} f(z_i)(x_i + x_{i-1})\Delta x_i$.

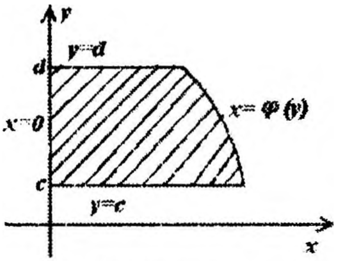
Bu yig'indi *[a:b]* kesmada uzluksiz $2\pi xf(x)$ funksiya uchun integral yig'indi bo'lganligi sababli $\max \Delta x_i \to 0$ da aniq limitga ega. $x_{i-1}<z_i<x_i$ bo'lib $\Delta x_i = x_i - x_{i-1}$ 0 ga intilganda $x_{i-1}+x_i$ yig'indi $2z_i$ ga intiladi.

Shunday qilib

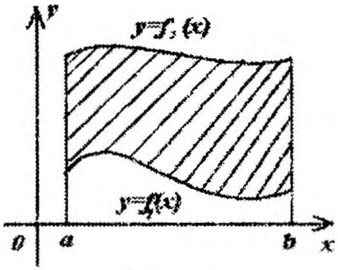
$$V_y = \lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} \pi f(z_i)(x_i + x_{i-1})\Delta x_i =$$

$$\lim_{\max \Delta x_i \to 0} \sum_{i=1}^{n} 2\pi f(z_i) z_i \Delta x_i = 2\pi \int_a^b f(x) x dx. \tag{43.2}$$

formulaga ega bo'lamiz.



179-chizma.



180-chizma.

Agar egri chiziqli trapetsiya $x=\varphi(y)$ egri chiziq, $0y$ o'q, $y=c$, $y=d$ $(c<d)$ to'g'ri chiziqlar bilan chegaralangan bo'lsa (179-chizma), u holda bu figuraning $0y$ o'q atrofida aylanishidan hosil bo'lgan jismning hajmi

$$V_y = \pi \int_c^d x^2 dy \tag{43.3}$$

formula yordamida hisoblanadi.

Agar $y=f_1(x)$, $y=f_2(x)$, $(f_1(x) \le f_2(x))$ egri chiziqlar va $x=a$ hamda $x=b$ to'g'ri chiziqlar bilan chegaralangan figura $0x$ o'q atrofida aylanayotgan bo'lsa (180-chizma), u holda hosil bo'lgan aylanish jismining hajmi

$$V_x = \pi \int_a^b [f_2^2(x) - f_1^2(x)] dx \tag{43.4}$$

formula yordamida hisoblanadi. Agar bu figura Oy o'q atrofida aylanayotgan bo'lsa, u holda aylanish jismining hajmi

$$V_y = \pi\int_a^b [f_2(x) - f_1(x)]x dx \qquad (43.5)$$

formula bo'yicha hisoblanadi.

Agar $y=f(x)$ egri chiziq parametrik yoki qutb koordinatalar sistemasida berilgan bo'lsa, u holda yuqorida keltirilgan barcha formulalarda integrallash o'zgaruvchisini tegishlicha almashtirish kerak.

Aylanish jismining hajmini va yon sirtini topishda quyida keltiriladigan teoremalaridan foydalanish ham mumkin.

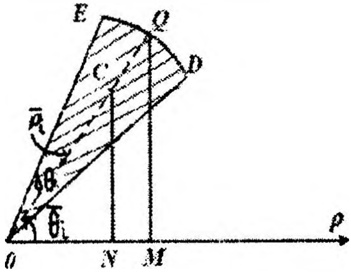
Guldenning birinchi teoremasi. Tekis egri chiziq yoyini uning tekisligida yotib, uni kesmaydigan birorta o'q atrofida aylanishidan hosil bo'lgan jism yon sirti yoy uzunligini bu egri chiziq yoyining og'irlik markazi chizadigan aylananing uzunligiga ko'paytmasiga teng.

Guldenning ikkinchi teoremasi. Tekis figurani uning tekisligida yotib, uni kesmaydigan birorta o'q atrofida aylanishidan hosil bo'lgan jismning hajmi bu figura yuzini uning og'irlik markazi chizadigan aylana uzunligiga ko'paytmasiga teng.

Endi Guldenning ikkinchi teoremasidan foydalanib uzluksiz $\rho = \varphi(\theta)$ egri chiziq hamda $\theta = \alpha, \theta = \beta$ nurlar bilan chegaralangan OAB egri chiziqli sektorning qutb o'qi ρ atrofida aylanishi natijasida hosil bo'lgan jismning hajmini topish formulasini chiqaramiz.

OAB egri chiziqli sektorni olib (165-chizma) uni xuddi egri chiziqli sektorning yuzini topishdagidek n ta qismlarga ajratamiz va i-doiraviy sektor ODE ni qaraymiz. Uni qutb o'qi ρ atrofida aylanishi natijasida hosil bo'lgan jismning hajmini topamiz. Bu sektorning yuzi

$\frac{1}{2}\bar{\rho}_i^2\Delta\theta_i$ $(\bar{\rho}_i = \varphi(\bar{\theta}_i) = OQ)$ bo'ladi. Doiraviy sektorning og'irlik markazini C orqali belgilasak u $\bar{\rho}_i = OQ$ kesmani $OC : CQ = 2:1$ nisbatda bo'ladi (chunki uchburchakning og'irlik markazi medianalari kesishgan nuqtada bo'lib medianalar kesishish nuqtasida uchburchakning uchidan hisoblaganda 2:1 nisbatda bo‘linadi).



181-chizma.

ΔOMQ dan $\dfrac{QM}{OQ}=\sin\overline{\theta}_i$, bundan $QM=\overline{\rho}_i\,\sin\overline{\theta}_i$.

ONC va OMQ uchburchakning o‘xshashligidan $\dfrac{CN}{OC}=\dfrac{QM}{OQ}$,

bundan $CN=\dfrac{OC}{OQ}\cdot QM=\dfrac{\frac{2}{3}OQ}{OQ}QM=\dfrac{2}{3}\overline{\rho}_i\,\sin\overline{\theta}_i$.

Guldenning ikkinchi teroemasiga ko‘ra bu i-sektorning ρ qutb o‘qi atrofida aylanishi natijasida hosil bo‘lgan jismning hajmi

$$dv_\rho=\frac{1}{2}\overline{\rho}_i^{\,2}\Delta\theta_i\cdot 2\pi CN=\frac{2}{3}\overline{\rho}_i^{\,3}\sin\overline{\theta}_i\Delta\theta_i=\frac{2}{3}\varphi^3(\overline{\theta}_i)\sin\overline{\theta}_i\Delta\theta_i$$

bo‘ladi, chunki og‘irlik markazi C nuqta ρ qutb o‘qi atrofida aylanish jarayonida radiusi CN ga teng aylanani chizadi.

Demak, OAB egri chiziqli sektorning qutb o‘qi atrofida aylanishi natijasida hosil bo‘lgan aylanish jismning hajmi

$$v_\rho\approx\frac{2}{3}\sum_{i=1}^{n}\varphi^3(\overline{\theta}_i)\sin\overline{\theta}_i\Delta\theta_i$$

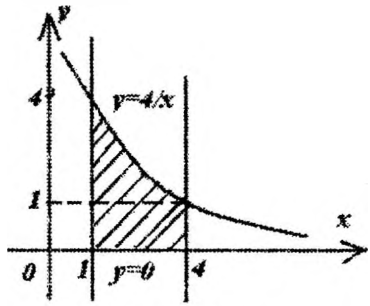
bo‘ladi.

Bunda $\max \Delta\theta_i \to 0$ limitga o'tib egri chiziqli sektorni qutb o'qi atrofida aylanishi natijasida hosil bo'lgan aylanish jismining hajmini topish uchun

$$v_\rho = \frac{2}{3}\int_\alpha^\beta \rho^3 \sin\theta d\theta = \frac{2}{3}\int_\alpha^\beta \varphi^3(\theta)\sin\theta d\theta \quad (43.6)$$

formulani hosil qilamiz.

1-misol. *xy=4, x=1, x=4, y=0* chiziqlar bilan chegaralangan figurani *0x* va *0y* o'qlari atrofida aylanishidan hosil bo'lgan jismning hajmini toping (182-chizma).



182-chizma.

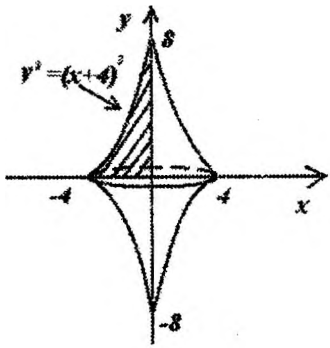
Yechish. (43.1) formulaga binoan:

$$V_{\delta} = \pi\int_1^4 \left(\frac{4}{x}\right)^2 dx = 16\pi\int_1^4 \left(\frac{1}{x^2}\right) dx = 16\pi\left(-\frac{1}{x}\right)\Big|_1^4 = 16\pi\left(-\frac{1}{4}+1\right) = 16\pi\cdot\frac{3}{4} = 12\pi.$$

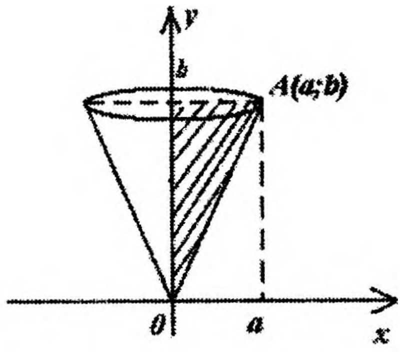
(43.2) formulaga binoan:

$$V_y = 2\pi\int_1^4 \frac{4}{x}\cdot x dx = 8\pi\int_1^4 dx = 8\pi\cdot x\Big|_1^4 = 8\pi(4-1) = 24\pi$$

2-misol. $y^2=(x+4)^3$ va $x=0$ chiziqlar bilan chegaralangan figura *0y* o'q atorofida aylanishidan hosil bo'lgan jismning hajmini toping (183-chizma).



183-chizma.



184-chizma.

Yechish. Hajmni (43.2) formuladan foydalanib hisoblaymiz, bunda $x\le 0$ ekanini nazarda to‘tamiz.

$$V_y = 2\cdot 2\pi\int_{-4}^{0} xy\,dx = -4\pi\int_{-4}^{0} x(x+4)^{\frac{3}{2}}dx = -4\pi\int_{-4}^{0}(x+4-4)(x+4)^{\frac{3}{2}}dx =$$

$$= -4\pi\int_{-4}^{0}(x+4)^{\frac{5}{2}}dx + 16\pi\int_{-4}^{0}(x+4)^{\frac{3}{2}}dx = -4\pi\frac{(x+4)^{\frac{7}{2}}}{7/2}\Big|_{-4}^{0} + 16\pi\frac{(x+4)^{\frac{5}{2}}}{5/2}\Big|_{-4}^{0} =$$

$$= -\frac{8\pi}{7}\cdot 4^{\frac{7}{2}} + \frac{32\pi}{5}\cdot 4^{\frac{5}{2}} = -\frac{8\pi}{7}\cdot 2^7 + \frac{32\pi}{5}\cdot 2^5 = \frac{1024\pi}{5} - \frac{1024\pi}{7} = \frac{2048\pi}{35}.$$

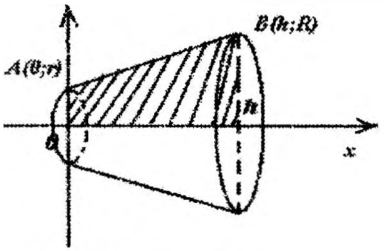
3-misol. Koordinatalar boshini $A(a;b)(a>0,\ b>0)$ nuqta bilan tutashtiruvchi to‘g‘ri chiziq kesmasi, *oy* o‘q hamda $y=b$ to‘g‘ri chiziq bilan chegaralangan figura *oy* o‘q atrofida aylanadi. Hosil bo‘lgan konusning hajmini hisoblang (184-chizma).

Yechish. $O(0;0)$ va $A(a;b)$ nuqtalardan o‘tuvchi to‘g‘ri chiziq tenglamasi $x=\frac{a}{b}y$ bo‘lishi ravshan. (43.3) formulaga binoan

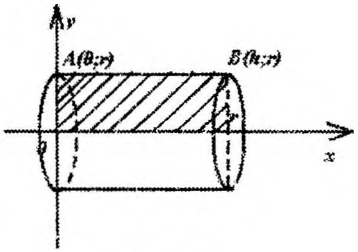
$$V_y = \pi\int_a^b x^2dy = \pi\int_0^b \frac{a^2}{b^2}y^2dy = \frac{\pi a^2}{b^2}\cdot\frac{y^3}{3}\Big|_0^b = \frac{1}{3}\pi a^2 b,$$

ya‘ni konusning hajmi asosining yuzi πa^2 bilan balandligi b ning ko'paytmasini uchdan biriga teng bo'lar ekan. Bu formula maktab kursidan bizga ayon.

4-misol. $A(0;r)$ va $B(h;R)$ nuqtalar berilgan, bunda $h>0$, $R\geq r>0$. AB kesma, $0x$ o'q hamda $x=0$, $x=h$ vertikal to'g'ri chiziqlar bilan chegaralangan figura $0x$ o'q atrofida aylanishi natijasida hosil bo'lgan jismning hajmini hisoblang.



185-chizma.



186-chizma.

Yechish. *a)* AB kesma $0x$ o'qqa parallel bo'lmasin, ya'ni $r\neq R$. U holda AB kesma $0x$ o'q atrofida aylanganda kesik konus hosil bo'ladi (185-chizma).

Ikki nuqtadan o'tuvchi to'g'ri chiziq tenglamasi

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}.$$

dan floydalanib $A(0;r)$ va $B(h;R)$ nuqtalrdan o'tuvchi to'g'ri chiziq tenglamasini topamiz:

$$\frac{x-0}{h-0}=\frac{y-r}{R-r} \quad \text{yoki} \quad y-r=\frac{R-r}{h}x, \quad y=r+\frac{R-r}{h}x.$$

(43.1) formulaga binoan quyidagini hosil qilamiz:

$$V_x=\pi\int_a^b y^2dx=\pi\int_0^h (r+\frac{R-r}{h}x)^2dx=\pi\int_0^h (r^2+2r\cdot\frac{R-r}{h}x+\left(\frac{R-r}{h}\right)^2x^2)dx=$$

$$=\pi[r^2x+r\cdot\frac{R-r}{h}x^2+\frac{(R-r)^2}{h^2}\cdot\frac{x^3}{3}]\Big|_0^h=\pi[r^2h+r\cdot(R-r)h+\frac{(R-r)^2}{3}h]=$$

$$=\frac{\pi h}{3}(3r^2+3rR-3r^2+R^2-2rR+r^2)=\frac{\pi h}{3}(R^2+rR+r^2).$$

Shunday qilib asoslarining radiuslari r, R va balandligi h bo'lgan kesik konusning hajmi

$$\vartheta = \frac{\pi h}{3}(R^2 + rR + r^2) \qquad (43.7)$$

formula yordamida topilar ekan.

b) AB kesma $0x$ o'qqa parallel ya'ni $r=R$ bo'lsin (186-chizma). U holda AB kesma $0x$ o'q atrofida aylanganda doiraviy silindr hosil bo'ladi. Qaralayotgan holda AB tenglamasi $y=r$ ko'rinishga ega bo'ladi. (43.1) formulaga binoan

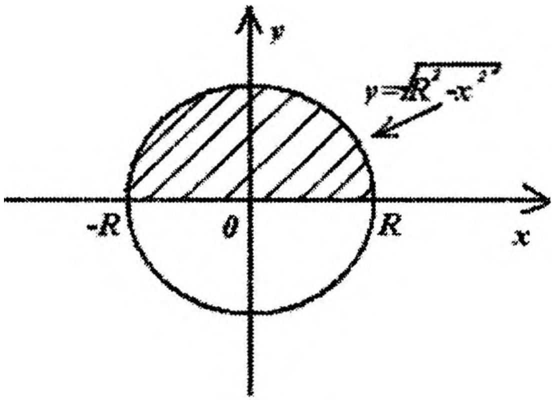
$$\vartheta_x = \pi\int_a^b y^2 dy = \pi\int_0^b r^2 dx = \pi r^2 h$$

bo'ladi. Demak asosining radiusi r, balandligi h bo'lgan doiraviy silindrning hajmi

$$\vartheta = \pi r^2 h$$

formula yordamida topilar ekan. Ya'ni to'g'ri doiraviy silindrning hajmi asosining yuzi bilan balandligining ko'paytmasiga teng ekan.

5-misol. $y=\sqrt{R^2-x^2}$ yarim aylana hamda $0x$ o'q bilan chegaralangan figura $0x$ o'q atrofida aylanadi. Hosil bo'lgan jismning hajmini hisoblang (187-chizma).



187-chizma.

Yechish. (43.1) formulaga binoan:

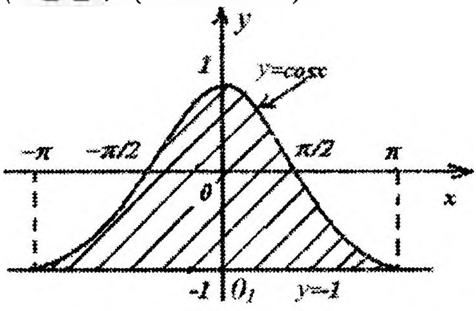
$$\vartheta_x = \pi\int_{-R}^{R}(R^2-x^2)dx = \pi(R^2x-\frac{x^3}{3})\Big|_{-R}^{R} = \pi(R^2\cdot R-\frac{R^3}{3})-\pi(-R^3+\frac{R^3}{3}) = \frac{4}{3}\pi R^3.$$

Agar biz $y=\sqrt{R^2-x^2}$ egri chiziq $0x$ o'q bilan chegaralangan soha $0x$ o'q atrofida aylanishi natijasida radiusi R ga teng shar hosil bo'lishini hisobga olsak sharning hajmini topish uchun

$$\vartheta = \frac{4}{3}\pi R^3$$

formulaga ega bo'lamiz.

6-misol. $y=\cos x$, $y=-1$ chiziqlar bilan chegaralangan figuraning $y=-1$ to'g'ri chiziq atrofida aylanishidan hosil bo'lgan jismning hajmini toping $(-\pi\le x\le\pi)$ (188-chizma).



188-chizma.

Yechish. $y_1=y+1$, $x_1=x$ almashtirish olsak koordinatalar boshi $0_1(0;-1)$ nuqtada bo'lgan ya'ni $0_1\ x_1y$ sistemaga ega bo'lamiz. Berilgan figura yangi sistemada yuqoridan $y=1+\cos x$ egri chiziq, quyidan $0_1\ x_1$ o'q bilan chegaralangan figuraga aylanadi.

Shuning uchun (43.1) formulaga binoan quyidagini hosil qilamiz:

$$\vartheta_x = \pi\int_a^b y^2dx = \pi\int_{-\pi}^{\pi}(1+\cos x)^2dx = \pi\int_{-\pi}^{\pi}(1+2\cos x+\cos^2 x)^2dx =$$

$$= \pi\int_{-\pi}^{\pi}(1+2\cos x+\frac{1+\cos 2x}{2})dx = \pi\left(x+2\sin x+\frac{x+\frac{1}{2}\sin 2x}{2}\right)\Bigg|_{-\pi}^{\pi} =$$

$$= \pi\left(\frac{3}{2}x+2\sin x+\frac{1}{4}\sin 2x\right)\Bigg|_{-\pi}^{\pi} = \pi\frac{3}{2}\cdot 2\pi = 3\pi^2.$$

7-misol. $x=a(t\text{-}\sin t)$, $y=a(1\text{-}\cos t)$ sikloidaning bir arkasi va $0x$ bilan chegaralangan figuraning: *a)* $0x$ o'q; *b)* $0y$ o'q; *c)* simmetriya o'qi atrofida aylanishidan hosil bo'lgan jismning hajmini toping (173-chizma).

Yechish. *a)* (43.1)formulaga binoan:

$$\vartheta_x = \pi\int_0^{2a\pi} y^2dx.$$

t parametrga o'tish uchun $x=a(t\text{-}\sin t)$, $y=a(1\text{-}\cos t)$, $dx=a(1\text{-}\cos t)dt$ formulalardan foydalanamiz; $x=0$ da $t=0$; $x=2\pi a$ da $t=2\pi$.

Demak

$$V_x = \pi\int_0^{2\pi} a^2(1-\cos t)^2 a(1-\cos t)dt = \pi a^3\int_0^{2\pi}(1-\cos t)^3dt =.$$

$$= \pi a^3\int_0^{2\pi}(1-3\cos t+3\cos^2 t-\cos^3 t)dt = \pi a^3\int_0^{2\pi}(1-3\cos t+$$

$$+\frac{3}{2}(1+\cos 2t)-(1-\sin^2 t))\cos t)dt = \pi a^3(t-3\sin t+\frac{3}{2}t+\frac{3}{2}\cdot\frac{1}{2}\sin 2t-$$

$$-\sin t++\frac{1}{3}\sin^3 t)\Big|_0^{2\pi} = \pi a^3(2\pi+\frac{3}{2}\cdot 2\pi) = 5\pi^2 a^3.$$

b) $0y$ o'q atrofida aylanishdan hosil bo'lgan jismning hajmini (43.2) formuldan foydalanib topamiz:

$$V_y = 2\pi \int_0^{2\pi a} xydx = .2\pi \int_0^{2\pi} a(t-\sin t)a(1-\cos t)a(1-\cos t)dt = .$$

$$= 2\pi a^3 \int_0^{2\pi} (t-\sin t)(1-\cos t)^2 dt = 2\pi a^3 \int_0^{2\pi} (t-\sin t)(1-2\cos t+\cos^2 t)dt = .$$

$$= 2\pi a^3 \int_0^{2\pi} (t-2t\cos t + t\cos^2 t - \sin t + \sin 2t - \sin t\cdot\cos^3 t)dt.$$

2-,4-,5- va *6-* qo'shiluvchilarning integrallarini nolga tengligini tekshirib ko'rish qiyin emas.

Shuning uchun quyidagi integrallarni hisoblash kifoya:

$$\int_0^{2\pi} (t + t\cos^2 t)dt = \int_0^{2\pi} (t + t\cdot\frac{1+\cos 2t}{2})dt = \int_0^{2\pi} \left(\frac{3}{2}t + \frac{1}{2}t\cos 2t\right)dt =$$

$$= \frac{3}{2}\cdot\frac{t^2}{2}\Big|_0^{2\pi} + \frac{1}{4}\int_0^{2\pi} td\sin 2t = \frac{3}{4}(2\pi)^2 + \frac{1}{4}t\cdot\sin 2t\Big|_0^{2\pi} - \frac{1}{4}\int_0^{2\pi}\sin 2tdt = 3\pi^2.$$

Demak, $V_y = 2\pi a^3 \cdot 3\pi^2 = 6\pi^3 \cdot a^3$.

b) Sikloidaning simmetriya o'qi $x = \pi a$ ko'rinishdagi tenglamaga ega. Agar $x_1 = x - \pi a$, $y_1 = y$ almashtirish olsak koordinatalar boshi $0_1(\pi a, 0)$ nuqtaga ko'chadi hamda $0\ y_1$ simmetriya o'qi bilan ustma-ust tushadi va (43.2) formuladan foydalanish imkoni tug'iladi.

$x = 2\pi a$ da $x_1 = a\pi$, $x = a\pi$ da $x_1 = 0$ bo'lishi ravshan.

Demak,

$$V_y = 2\pi\int_a^b xydx = 2\pi\int_0^{a\pi} x_1 y_1 dx_1 = 2\pi\int_{a\pi}^{2a\pi}(x-a\pi)yd(x-a\pi) =$$

$$= 2\pi\int_{a\pi}^{2a\pi}(x-a\pi)ydx = 2\pi\int_{\pi}^{2\pi}[a(t-\sin t)-a\pi]a(1-\cos t)a(1-\cos t)dt =$$

$$= 2\pi a^3\int_{\pi}^{2\pi}(t-\sin t-\pi)(1-\cos t)^2 dt.$$

Qavslarni ochib va tegishli integrallarni hisoblab, uzil-kesil topamiz:

$$V_{y_1}=2\pi a^3\left(\frac{3}{4}\pi^2-\frac{4}{3}\right)=\frac{\pi a^3}{6}(9\pi^2-16).$$

8-misol. $\rho=a(1+\cos\theta)$ kardioida hamda qutb o'qi bilan chegaralangan figuraning qutb o'qi atrofida aylanishi natijasida hosil bo'lgan jismning hajmini hisoblang (166-chizma).

Yechish. Kardiolida qutb o'qiga nisbatan simmetrik va θ qutb burchagi 0 dan π gacha o'zgarganda kardioidaning qutb o'qining yuqorisida joylashgan yarmi chiziladi.

$$V_\rho=\frac{2}{3}\pi a^3\int_0^\pi(1+\cos\theta)^3\sin\theta d\theta=\frac{2}{3}\pi a^3\int_0^\pi(\sin\theta+$$

$$+3\sin\theta\cos\theta+3\cos^2\theta\sin\theta+\cos^3\theta\sin\theta)d\theta=$$

$$=\frac{2}{3}\pi a^3\left[\int_0^\pi\sin\theta d\theta+\frac{3}{2}\int_0^\pi\sin 2\theta d\theta-3\int_0^\pi\cos^2\theta d\cos\theta-\int_0^\pi\cos^3\theta d\cos\theta\right]=$$

$$\frac{2}{3}\pi a^3(-\cos\theta-\frac{3}{4}\cos 2\theta-3\cdot\frac{\cos^3\theta}{3}-\frac{\cos^4\theta}{4})\Big|_0^\pi=\frac{2}{3}\pi a^3(2+2)=\frac{8\pi a^3}{3}.$$

Shuning uchun (43.6) formulaga ko'ra quyidagini hosil qilamiz.

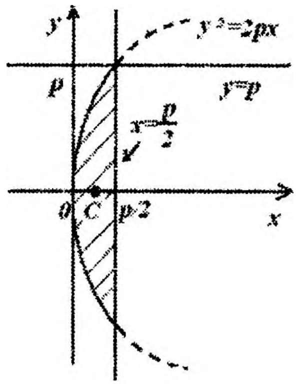
9-misol. $x^2+y^2\le a^2$ doiraning $x=b>a$ to'gri chiziq atrofida aylanishi natijasida hosil bo'lgan tor deb ataluvchi jismning hajmi topilsin.

Yechish. Doiraning markazi uning og'irlik markazihamdir. Doiraning markazi koordinatalar boshida bo'lganligi sababli u $x=b$ to'g'ri chiziq atrofida aylanganda uzunligi $2\pi b$ ga teng aylana chizadi. Doiraning yuzi πa^2 ekanligi ma'lum. Shuning uchun Guldenning ikkinchi teoremasiga binoan

$$V=\pi a^2\cdot 2\pi b=2\pi^2a^2b$$

formulaga ega bo'lamiz.

10-misol. $y^2=2px$ parabola va $x=\frac{p}{2}$ to'g'ri chiziq bilan chegaralangan parabolik segmentning $y=p$ to'g'ri chiziq atrofida aylanishidan hosil bo‘lgan jismning hajmini hisoblang.



189-chizma.

Yechish. Parabolik segment $0x$ o‘qqa nisbatan simmetrik bo‘lganligi uchun og‘irlik markazi C abssissalar o‘qida joylashgan. C nuqta $y=p$ to‘g‘ri chiziq atrofida aylanganda uzunligi $2\pi p$ ga teng aylana chizadi.

Parabolik segmentning yuzi

$$Q = 2\int_0^{\frac{p}{2}} \sqrt{2px}\,dx = \frac{2}{3}p^2$$

ga teng. Shuning uchun Guldenning ikkinchi teoremasiga ko‘ra

$$V = \frac{2}{3}p^2 \cdot 2\pi p = \frac{4}{3}\pi p^3$$

formulaga ega bo‘lamiz.

43.2. Aylanish jismining sirti

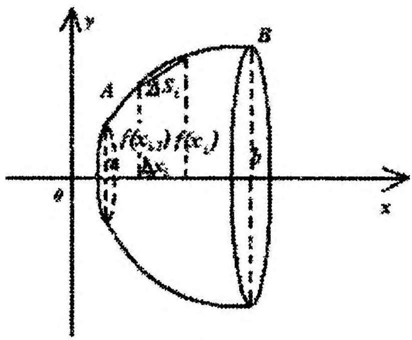
Faraz qilaylik $y=f(x)$ funksiya $[a;b]$ kesmada uzluksiz va uzluksiz $f'(x)$ hosilaga ega bo‘lsin.

$y=f(x)$ egri chiziqning $x=a$, $x=в$ vertikal to‘g‘ri chiziqlar orasidagi AB yoyining $0x$ o‘q atrofida aylanishi natijasida hosil bo‘lgan jismning yon sirtini topish talab etilsin. $[a;b]$ kesmani

$$a=x_0< x_1< x_2<...< x_{n-1}< x_n=b$$

bo‘linish nuqtalari yordamida n ta ixtiyoriy bo‘laklarga bo‘lamiz. AB egri chiziqda abssissalari $x_0,x_1,x_2,...,x_{n-1},x_n$ nuqtalardan iborat

$A, M_1, M_2,..., M_{i-1}, M_i,..., M_{n-1}, B$ nuqtalarni olib AM_1, $M_1M_2,...,$ $M_{i-1}M_i,...,M_{n-1}B$ vatarlarni o‘tkazamiz va ularning uzunliklarini mos ravishda $\Delta S_1,\Delta S_2,...,\Delta S_i,...,\Delta S_n$ lar orqali belgilaymiz. U holda AB yoyga ichki chizilgan $AM_1M_2...M_{i-1}\, M_i...M_{n-1}B$ siniq chiziq hosil bo‘ladi. Har qaysi uzunligi ΔS_i $(i=\overline{1,n})$ bo‘lgan $M_{i-1}\, M_i$ vatar $0x$ o‘q atrofida aylanishi natijasida kesik konus (yoki silindr) hosil bo‘ladi. Bu kesik konusning asoslarining radiuslari $y_{i-1}= f(x_{i-1})$, $y_i= f(x_i)$ bo‘lishi ravshan. Kesik konusning yon sirti asos aylanalari uzunliklari yig‘indisining yarmi bilan yasovchisi ko‘paytmasiga teng edi. Shunga ko‘ra i-kesik konusning yon sirti



190-chizma.

$$\Delta P_i = 2\pi \frac{f(x_{i-1}) + f(x_i)}{2} \Delta S_i$$

bo'ladi. Yoy uzunligini topishda

$$\Delta S_i = \sqrt{1+[f'(z_i)]^2}\,\Delta x_i, \quad x_{i-1} < z_i < x_i$$

ekanligiga iqror bo'lgan edik. Buni e'tiborga olsak

$$\Delta P_i = \pi[f(x_{i-1}) + f(x_i)]\sqrt{1+[f'(z_i)]^2}\,\Delta x_i$$

bo'ladi.

AB yoyga ichki chizilgan siniq chiziqning *0x* o'q atrofida aylanishi natijasida hosil bo'lgan jismning yon sirti

$$P_n = \pi\sum_{i=i}^{n}[f(x_{i-1}) + f(x_i)]\sqrt{1+[f'(z_i)]^2}\,\Delta x_i$$

yig'indiga teng bo'ladi. Siniq chiziqning eng katta bo'g'ini (zvenosi)ning uzunligi ΔS_i nolga intilgandagi bu yig'indining limiti qaralayotgan aylanish jismining sirti deyiladi.

$$P = \lim_{\max \Delta S_i \to 0} P_n = \lim_{\max \Delta x_i \to 0} P_n = \lim_{\max \Delta x_i \to 0} \pi\sum_{i=i}^{r}[f(x_{i-1}) + f(x_i)]\sqrt{1+[f'(z_i)]^2}\,\Delta x_i =$$

$$= \pi \lim_{\max \Delta x_i \to 0} \sum_{i=i}^{r} 2f(z_i)\sqrt{1+[f'(z_i)]^2}\,\Delta x_i = 2\pi\int_a^b f(x)\sqrt{1+f'^2(x)}\,dx.$$

Shunday qilib aylanish jismining yon sirti

$$P_x = 2\pi\int_a^b y\sqrt{1+y'^2}\,dx = 2\pi\int_a^b f(x)\sqrt{1+f'^2(x)}\,dx. \quad (43.7)$$

formula yordamida topilar ekan.

Eslatma. Aylanish jismining to'la sirtini topish talab etilganda uning yon sirtiga asoslarining radiuslari *f(a)* va *f(b)* bo'lgan doiralarning yuzlari qo'shiladi.

Agar $x = \varphi(y)$ egri chiziqning $y=c$, $y=d$ $(c<d)$ gorizontal to'g'ri chiziqlar orasidagi yoyining *0y* o'q atrofida aylanishi natijasida hosil bo'lgan jismning yon sirtini topish talab etilsa (179-chizma) u

$$P_y = 2\pi\int_c^d x\sqrt{1+x'^2}\,dy = 2\pi\int_c^d \varphi(y)\sqrt{1+\varphi'^2(y)}\,dy \quad (43.7')$$

formula yordamida topiladi, bunda $x = \varphi(y)$ egri chiziq silliq, ya'ni

$\varphi(y)$ funksiya $[c, d]$ kesmada uzluksiz va uzluksiz $\varphi'(y)$ hosilaga ega deb faraz qilinadi.

Agar egri chiziq $x=\varphi(t)$, $y=\psi(t)$ $\alpha\leq t\leq\beta$ parametrik tenglamalari yordamida berilgan bo'lib $\varphi(t)$, $\psi(t)$ funksiyalar $[\alpha, \beta]$ kesmada uzluksiz va uzluksiz hosilalarga ega bo'lsa hamda $\varphi(\alpha)=a$, $\psi(\beta)=b$ bo'lsa, u holda (43.7) formulada $x=\varphi(t)$, almashtirish olib aylanish jismining yon sirtini topish uchun

$$P_x = 2\pi\int_{\alpha}^{\beta}\psi(t)\sqrt{\varphi'^2(t)+\psi'^2(t)}dt \qquad (43.8)$$

formulani hosil qilamiz.

Egri chiziq qutb koordinatalar sistemasida $\rho=f(\theta)$, $\alpha\leq\theta\leq\beta$ tenglama bilan berilgan bo'lib egri chiziqli sektorni ρ qutb o'qi atrofida aylanishi natijasida hosil bo'lgan jismning sirti yuzini topish talab etilsin. Agar $f(\theta)$ funksiya $[\alpha, \beta]$ kesmada uzluksiz va uzluksiz $f'(\theta)$ hosilaga ega bo'lsa egri chiziqni $x=f(\theta)\cos\theta$, $y=f(\theta)\sin\theta$ parametrik tenglamalari yordamida berilgan deb qarash mumkin. Shuning uchun bu holda aylanish jismining yon sirti

$$P_\rho = 2\pi\int_{\alpha}^{\beta} f(\theta)\sin\theta\sqrt{f^2(\theta)+f'^2(\theta)}d\theta. \qquad (43.9)$$

formula yordamida topiladi.

11-misol. $y=\sqrt{R^2-x^2}$ yarim aylanani Ox o'q atrofida aylanishi natijasida hosil bo'lgan sferaning sirtini hisoblang.

Yechish.

$$y'=(\sqrt{R^2-x^2})'=-\frac{x}{\sqrt{R^2-x^2}};\quad \sqrt{1+y'^2}=\sqrt{1+\frac{x^2}{R^2-x^2}}=\frac{R}{\sqrt{R^2-x^2}}.$$

$$P_x=2\pi\int_{-R}^{R}\sqrt{R^2-x^2}\cdot\frac{R}{\sqrt{R^2-x^2}}dx=2\pi Rx\Big|_{-R}^{R}=4\pi R^2.$$

12-misol. $x=a(t-\sin t)$, $y=a(1-\cos t)$, $0\leq t\leq 2\pi$ sikloida bitta arkasini

Ox o'q atrofida aylanishi natijasida hosil bo'lgan jismning sirtini hisoblang.

Yechish. $\sqrt{x'^2+y'^2}=2a\sin\frac{t}{2}$ edi (42-mavzu 5-misolga qarang).

Shuning uchun (43.8) formulaga binoan quyidagiga ega bo'lamiz.

$$P_x=2\pi\int_0^{2\pi}a(1-\cos t)2a\sin\frac{t}{2}dt=.8\pi a^2\int_0^{2\pi}\sin^2\frac{t}{2}\cdot\sin\frac{t}{2}dt=.$$

$$=8\pi a^2\int_0^{2\pi}(1-\cos^2\frac{t}{2})(-2)d(\cos\frac{t}{2})dt=-16\pi a^2\left[\cos\frac{t}{2}-\frac{\cos^3\frac{t}{2}}{3}\right]\Bigg|_0^{2\pi}$$

$$=-16\pi a^2(-2+\frac{2}{3})=\frac{64}{3}\pi a^2.$$

13-misol. $\rho=a(1+\cos\theta)$ kardiodaning qutb o'qi atrofida aylanishidan hosil bo'lgan jismning sirtning yuzini toping (166-chizma).

Yechish. (43.9) formulaga binoan:

$$P_p=2\pi\int_0^{\pi}\rho\sin\theta\sqrt{\rho^2+\rho'^2}d\theta=$$

$$=2\pi\int_0^{\pi}a(1+\cos\theta)\sin\theta\sqrt{[a(1+\cos\theta)]^2+(a\sin\theta)^2}d\theta=$$

$$=2\pi a^2\int_0^{\pi}(1+\cos\theta)\sin\theta\cdot 2\cdot\cos\frac{\theta}{2}d\theta=4\pi a^2\int_0^{\pi}2\cos^2\frac{\theta}{2}\cdot 2\sin\frac{\theta}{2}\cos^2\frac{\theta}{2}d\theta=$$

$$=16\pi a^2\int_0^{\pi}\cos^4\frac{\theta}{2}\cdot\sin\frac{\theta}{2}d\theta=16\pi a^2\int_0^{\pi}\cos^4\frac{\theta}{2}(-2)d(\cos\frac{\theta}{2}).=$$

$$=-32\pi a^2\frac{\cos^5\frac{\theta}{2}}{5}\Bigg|_0^{\pi}=\frac{32\pi a^2}{5}.$$

14-misol. $y=\frac{x^2}{2}$ egri chiziq $y=1,5$ to'g'ri chiziq bilan kesilgan qismini Oy o‘q atrofida aylanishidan hosil bo‘lgan sirt yuzini hisoblang.

Yechish. Egri chiziq (parabola)ning birinchi chorakdagi qismi uchun

$$x = \sqrt{2y},\ x' = \frac{1}{\sqrt{2y}},\ \sqrt{1+x'^2} = \sqrt{1+\frac{1}{2y}}.$$

(43.7′) formulaga asosan

$$P_y = 2\pi\int_0^{1,5}\sqrt{2y}\cdot\sqrt{1+\frac{1}{2y}}dy = 2\pi\int_0^{1,5}\sqrt{2y+1}dy =$$

$$2\pi\cdot\frac{1}{2}\cdot\frac{(2y+1)^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_0^{1,5} = \frac{2}{3}\pi(8-1) = \frac{14}{3}\pi.$$

15-misol. $y = e^{-x}$ chiziqni x ning manfiy bo‘lmagan qiymatlariga mos keluvchi cheksiz yoyini abssissalar o‘qi atrofida aylanishidan hosil bo‘lgan sirt yuzini hisoblang (191-chizma).

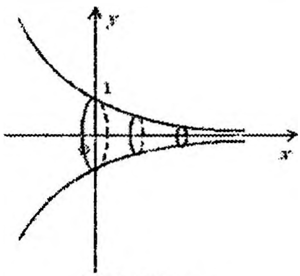
Yechish. $y = e^{-x}$ ga egamiz. Shuning uchun

$$y' = -e^{-x}, \sqrt{1+y'^2} = \sqrt{1+e^{-2x}}$$

bo‘lib (43.7) formulaga binoan

$$P_x = 2\pi\int_0^{+\infty}e^{-x}\sqrt{1+e^{-2x}}dx = -2\pi\int_0^{+\infty}\sqrt{1+e^{-2x}}d(e^{-x})$$

xosmas integralga ega bo‘ldik.



191-chizma.

$e^{-x} = t$ desak $x=0$ da $t=1$, $x \to +\infty$ da $t=0$ bo‘lib

$$P_x = -2\pi\int_1^0 \sqrt{1+t^2}\,dt = 2\pi\int_0^1 \sqrt{1+t^2}\,dt$$

bo‘ladi.

$$\int_0^1\sqrt{1+t^2}\,dt=\int_0^1\frac{1+t^2}{\sqrt{1+t^2}}dt=\int_0^1\frac{dt}{\sqrt{1+t^2}}+\int_0^1 t\cdot\frac{tdt}{\sqrt{1+t^2}}=\ln\left(t+\sqrt{1+t^2}\right)\Big|_0^1+$$

$$+\int_0^1 t\cdot\frac{tdt}{\sqrt{1+t^2}}=\ln\left(t+\sqrt{1+t^2}\right)\Big|_0^1+\int_0^1 td\left(\sqrt{1+t^2}\right)=\ln(1+\sqrt{2})+t\cdot\sqrt{1+t^2}\Big|_0^1-$$

$$-\int_0^1\sqrt{1+t^2}\,dt=\ln(1+\sqrt{2})+\sqrt{2}-\int_0^1\sqrt{1+t^2}\,dt.$$

Bu yerda bo‘laklab integrallash forfulasidan foydalanildi.

So‘nggi integralni tenglikning chap tomoniga o‘tkazsak

$$2\int_0^1\sqrt{1+t^2}\,dt = \ln(1+\sqrt{2})+\sqrt{2}$$

va bundan

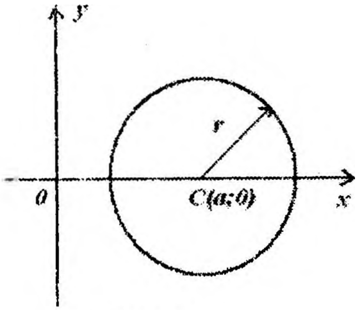
$$\int_0^1\sqrt{1+t^2}\,dt = \frac{1}{2}[\ln(1+\sqrt{2})+\sqrt{2}]$$

kelib chiqadi.

Shunday qilib $P_x = \pi[\ln(1+\sqrt{2})+\sqrt{2}]$ bo‘lar ekan.

16-misol. $(x-a)^2+y^2=r^2$, $0<r<a$ aylanani Oy o‘q atrofida aylanishidan hosil bo‘lgan (bunaqa sirt tor deyiladi) sirt yuzini hisoblang (192-chizma).

Yechish. Aylananing og‘irlik markazi uning markazida bo‘ladi. Shuning uchun $(a;0)$ og‘irlik markazi Oy o‘q atrofida aylanganda uzunligi $2\pi a$ ga teng aylana chizadi. r radiusli aylananing uzunligi $2\pi r$ bo‘ladi.



192-chizma.

Shuning uchun Guldenning birinchi teoremasiga binoan torning sirti $P = 2\pi r \cdot 2\pi a = 4\pi^2 ra$ bo'ladi.

O'z-o'zini tekshirish uchun savollar

1. $y=f(x)$, $y=0$, $x=a$, $x=b$ chiziqlar bilan chegaralangan figuraning $0x$ va $0y$ o'qlar atrofida aylanishi natijasida hosil bo'lgan jismlarning hajmi qanday topiladi?
2. $x=\varphi(y)$, $x=0$, $y=c$, $y=d$ $(c<d)$ chiziqlar bilan chegaralangan figuraning $0y$ o'q atrofida aylanishi natijasida hosil bo'lgan jismning hajmi qanday topiladi?
3. $y=f_1(x)$, $y=f_2(x)$ $(f_1(x) \le f_2(x))$ $x=a$, $x=b$ $(a<b)$ chiziqlar bilan chegaralangan figurani $0x$ hamda $0y$ o'qlari atrofida aylanishi natijasida hosil bo'lgan jismlarning hajmlari qanday topiladi?
4. Qutb koordinatalar sistemasida berilgan egri chiziqli sektorni qutb o'qi atrofida aylanishidan hosil bo'lgan jismning hajmi qanday topiladi?
5. $y=f(x)$ egri chiziqni AB yoyini $0x$ o'q atrofida aylanishi natijasida hosil bo'lgan jismni yon sirti qanday topiladi?
6. Egri chiziq parametrik tenglamalari yordamida berilganda aylanish jismining yon sirti qanday topiladi?
7. Qutb koordinatalar sistemasida berilgan egri chiziqli sektorni qutb o'qi atrofida aylanishidan hosil bo'lgan jismning yon sirti qanday topiladi?
8. Shar sirtini topish formulasini keltirib chiqaring.
9. Konusning hajmini topish formulasini keltirib chiqaring.
10. Konusning yon sirtini topish formulasini keltirib chiqaring.

11. Kesik konusning hajmi va yon sirtini topish formulalarini keltirib chiqaring.

12. Silindrning hajmi va yon sirtini topish formulalarini keltirib chiqaring.

13. Guldenning teoremalarini ayting.

Mustaqil yechish uchun mashqlar

1. $y=x^2+3$, $x=0$, $x=4$, $y=0$ chiziqlar bilan chegaralangan figuraning $0x$ o'q atrofida aylanishidan hosil bo'lgan jismning hajmini toping. Javob: $368{,}8\,\pi$.

2. $y=2x-x^2$ va $0x$ o'q bilan chegaralangan figuraning $0x$ o'q atrofida aylanishidan hosil bo'lgan jismning hajmini toping. Javob: $\frac{16}{15}\pi$.

3. $xy=9$, $y=3$, $y=9$, $x=0$ chiziqlar bilan chegaralangan figuraning $0y$ o'q atrofida aylanishidan hosil bo'lgan jismning hajmini toping. Javob: $18\,\pi$.

4. $y=10-x^2$ va $y=x^2+2$ parabolalar hamda $0y$ o'q bilan chegaralangan figuraning $0y$ o'q atrofida aylanishidan hosil bo'lgan jismning hajmi topilsin. Javob: $16\,\pi$

5. $y=4-x^2$ parabola hamda $2x+y-4=0$ to'g'ri chiziq bilan chegaralangan figuraning $0x$ o'q atrofida aylanishidan hosil bo'lgan jismning hajmini toping. Javob: $\frac{32\pi}{5}$.

6. $x=a\cos^3 t$, $y=a\sin^3 t$ astroidaning $0y$ o'q atrofida aylanishidan hosil bo'lgan jismning hajmini toping (162-chizma) Javob: $\frac{32\,\pi a^3}{105}$.

7. $\rho=a\cos^2\theta$ egri chiziq hamda qutb o'qi bilan chegaralangan figurani qutb o'qi atrofida aylanishidan hosil bo'lgan jismning hajmini topilsin. Javob: $\frac{4}{21}\pi a^3$.

8. $x^2=4+y$ egri chiziq va $y=-2$ to'g'ri chiziq bilan chegaralangan figurani $0x$ o'q atrofida aylanishidan hosil bo'lgan jismning sirtini

hisoblang. Javob: $\frac{34\sqrt{17}-2}{9}\pi$.

9. $\rho=2a\sin\theta$ aylanani qutb o'qi atrofida aylanishidan hosil bo'lgan jismning sirtini toping. Javob: $4\pi^2a^2$

10. $x=a\cos^3t$, $y=a\sin^3t$ astroidaning abssissalar o'qi atrofida aylanishidan hosil bo'lgan jismning sirtini toping. Javob: $\frac{6\pi a^2}{5}$.

44. ANIQ INTEGRALNING MEXANIKAGA TATBIQLARI

44.1.Statik moment

m massali moddiy nuqtaning ℓ o'qqa nisbatan **statik momenti** deb $M_e=md$ kattalikka aytiladi, bu yerda d nuqtadan ℓ o'qqacha bo'lgan masofa.

Agar *oxy* tekislikda massalari $m_1, m_2, \ldots, m_i, \ldots m_n$ bo'lgan moddiy nuqtalarning

$$P_1(x_1, y_1),\ P_2(x_2, y_2), \ldots,\ P_i(x_i, y_i), \ldots,\ P_n(x_n, y_n) \qquad (44.1)$$

sistemasi berilgan bo'lsa , u holda $x_i\, m_i$ va $y_i\, m_i$ ko'paytmalar m_i massaning *oy* va *ox* o'qlarga nisbatan **statik momentlari** deyiladi. Berilgan (44.1) moddiy nuqtalar sistemasining og'irlik markazi koordinatalari

$$x_c = \frac{x_1 m_1 + x_2 m_2 + \ldots + x_n m_n}{m_1 + m_2 + \ldots + m_n} = \frac{\sum\limits_{i=1}^{n} x_i m_i}{\sum\limits_{i=1}^{n} m_i}, \qquad (44.2)$$

$$y_c = \frac{y_1 m_1 + y_2 m_2 + \ldots + y_n m_n}{m_1 + m_2 + \ldots + m_n} = \frac{\sum\limits_{i=1}^{n} y_i m_i}{\sum\limits_{i=1}^{n} m_i} \qquad (44.3)$$

formulalar yordamida topilishi mexanika kursidan ma'lum.

$$M_x=, \sum_{i=1}^{n} y_i m_i \qquad M_y=\sum_{i=1}^{n} x_i m_i \qquad (44.4)$$

yig'indilar berilgan sistemaning mos ravishda *oy* va *ox* o'qlarga nisbatan **statik momenti** deyiladi,

Agar moddiy nuqtalar sistemasi *ox* yoki *oy* o'qqa nisbatan simmetrik bo'lsa, u holda uning mos statik momentlari nolga teng bo'ladi.

44.2. Inersiya momenti

m massali moddiy nuqtaning ℓ o'qqa nisbatan inersiya momenti deb $I_\ell=md^2$ songa aytiladi, bu yerda d- nuqtadan o'qqacha masofa.

(44.1) moddiy nuqtalar sistemasining ox va oy o'qqa nisbatan inersiya momentlari

$$I_x=\sum_{i=1}^{n} y_i^2 m_i, \qquad I_y=\sum_{i=1}^{n} x_i^2 m_i \tag{44.5}$$

formulalar yordamida topiladi.

(44.2), (44.3),(44.4) va (44.5) formulalardan geometrik figura va jismlarning og'irlik markazini, statik va inersiya momentlarini topishda foydalanamiz.

44.3. Tekislikdagi chiziqning og'irlik markazi va statik hamda inersiya momentlari

Agar Oxy tekislikning $C(x_C;y_C)$ nuqtasiga massasi AB egri chiziqning massasiga teng moddiy nuqta joylashtirilganda bu nuqtaning istalgan koordinata o'qiga nisbatan statik momenti egri chiziqning shu o'qqa nisbatan statik momentiga son qiymat jihatdan teng bo'lsa, u holda $C(x_C;y_C)$ nuqta AB egri chiziqning og'irlik markazi deb ataladi.

Uzluksiz AB egri chiziq $y=f(x)$, $a\leq x\leq b$ tenglama bilan berilgan bo'lsin.

Berilgan chiziq uzunlik birligining massasi **chiziqli zichlik** deyiladi. Egri chiziqning hamma joyida chiziqli zichlik bir xil va γ ga teng (egri chiziq bir jinsli) deb faraz qilamiz.

AB egri chiziqni uzunliklari $\Delta s_1, \Delta s_2, \ldots \Delta s_n$ bo'lgan ixtiyoriy n ta bo'lakka bo'lamiz. Bu bo'laklarning massalari ularning uzunliklari bilan chiziqli zichlik ko'paytmasiga teng, ya'ni Δs_i $(i=\overline{1,n})$ ning massasi $\Delta m_i=\gamma\cdot\Delta s_i$ bo'ladi. AB yoyining har bir Δs_i bo'lagida abssissasi z_i bo'lgan $P_i\ (z_i, f(z_i))$ nuqta olamiz. AB yoyning har bir Δs_i bo'lagini massasi $\gamma\Delta s_i$ bo'lgan $P_i\ (z_i, f(z_i))$ moddiy nuqta deb qarab (44.2) va (44.3) formulalarda x_i o'rniga z_i ,y_i o'rniga $f(z_i)$,m_i o'rniga $\Delta m_i=\gamma\Delta s_i$ qiymatlarni qo'ysak, AB yoyining og'irlik markazini aniqlash uchun quyidagi taqribiy formulalarni hosil qilamiz:

$$x_c \approx \frac{\sum_{i=1}^{n} z_i \gamma \Delta s_i}{\sum_{i=1}^{n} \gamma \Delta s_i}, y_c \approx \frac{\sum_{i=1}^{n} f(z_i)\gamma \Delta s_i}{\sum_{i=1}^{n} \gamma \Delta s_i}.$$

y=f(x) *[a,b]* kesmada uzluksiz va uzluksiz *f'(x)* hosilaga ega bo'lsin. U holda $\Delta s_i = \sqrt{1+[f(z_i)]^2}\Delta x_i$ ekanligi ko'rsatilgan edi. Shuning uchun

$$x_c \approx \frac{\sum_{i=1}^{n} z_i \sqrt{1+[f'(z_i)]^2}\Delta x_i}{\sum_{i=1}^{n} \sqrt{1+[f'(z_i)]^2}\Delta x_i}, y_c \approx \frac{\sum_{i=1}^{n} f(z_i)\sqrt{1+[f'(z_i)]^2}\Delta x_i}{\sum_{i=1}^{n} \sqrt{1+[f'(z_i)]^2}\Delta x_i}$$

ga ega bo'lamiz.

Bu formulalarda $\max \Delta x_i \to o$ da limitga o'tib, *AB* yoyning og'irlik markazi koordinatalarini topish uchun quyidagi aniq formulalarni hosil qilamiz:

$$x_C = \frac{\int_a^b x\sqrt{1+y'^2}dx}{\int_a^b \sqrt{1+y'^2}dx} \quad (44.2'), \qquad y_C = \frac{\int_a^b y\sqrt{1+y'^2}dx}{\int_a^b \sqrt{1+y'^2}dx}. \quad (44.3')$$

Bu formulalardan *AB* yoyni statik momentini hisoblash uchun

$$M_x = \int_a^b y\sqrt{1+y'^2}dx, \ M_y = \int_a^b x\sqrt{1+y'^2}dx \qquad (44.4')$$

formulalarga ega bo'lamiz.

AB yoyning inersiya momentlarini hisoblash uchun quyidagi formulalarni ham xuddi shu usulda hosil qilish mumkin:

$$I_x = \int_a^b y^2\sqrt{1+y'^2}dx, \quad I_y = \int_a^b x^2\sqrt{1+y'^2}dx \qquad (44.5')$$

(44.2′) va (44.3′) formulalarning maxrajidagi ifoda AB yoyning massasi M ni ifodalashini ta'kidlab o'tamiz.

AB egri chiziq $x=\varphi(t)$, $y=\psi(t)$, $\alpha\leq t\leq\beta$ tenglamalar yordamida berilgan bo'lib $\varphi(t)$, $\psi(t)$ funksiyalar $[\alpha,\beta]$ kesmada uzluksiz va uzluksiz hosilalarga ega hamda $\varphi(\alpha)=a$, $\varphi(\beta)=b$ bo'lsa, u holda yuqorida chiqarilgan formulalarda $x=\varphi(t)$ almashtirish olib quyidagilarni hosil qilamiz:

$$x_C = \frac{\int_\alpha^\beta \varphi(t)\sqrt{[\varphi'(t)]^2+[\psi'(t)]^2}dt}{\int_\alpha^\beta \sqrt{[\varphi'(t)]^2+[\psi'(t)]^2}dt} \qquad (44.2''),$$

$$y_C = \frac{\int_\alpha^\beta \psi(t)\sqrt{[\varphi'(t)]^2+[\varphi'(t)]^2}dt}{\int_\alpha^\beta \sqrt{[\varphi'(t)]^2+[\varphi'(t)]^2}dt} \qquad (44.3'')$$

$$M_x = \int_\alpha^\beta \psi(t)\sqrt{[\varphi'(t)]^2+[\psi'(t)]^2}dt, \quad M_y = \int_\alpha^\beta \varphi(t)\sqrt{[\varphi'(t)]^2+[\psi'(t)]^2}dt \qquad (44.4'')$$

$$I_x = \int_\alpha^\beta \psi^2(t)\sqrt{\varphi'(t)]^2+[\psi'(t)]^2}dt, \quad I_y = \int_\alpha^\beta \varphi^2(t)\sqrt{[\varphi'(t)]^2+[\psi'(t)]^2}dt \qquad (44.5'')$$

AB egri chiziq qutb koordinatalar sistemasida $\rho=f(\theta)$, $\alpha\leq\theta\leq\beta$ tenglama bilan berilgan bo'lib $f(\theta)$ funksiya $[\alpha,\beta]$ kesmada uzluksiz va uzluksiz $f'(\theta)$ hosilaga ega bo'lsin. U holda AB yoy og'irlik markazining abssissasi va ordinatasi

$$x_c = \frac{\int\limits_{\alpha}^{\beta} \rho \cos\theta \sqrt{\rho^2 + \rho'^2}\, d\theta}{\int\limits_{\alpha}^{\beta} \sqrt{\rho^2 + \rho'^2}\, d\theta} \qquad (44.2'''),$$

$$y_c = \frac{\int\limits_{\alpha}^{\beta} \rho \sin\theta \sqrt{\rho^2 + \rho'^2}\, d\theta}{\int\limits_{\alpha}^{\beta} \sqrt{\rho^2 + \rho'^2}\, d\theta} \qquad (44.3''')$$

formulalar yordamida topiladi.

AB yoyni qutb o'qiga nisbatan statik momenti

$$M_\rho = M_x = \int\limits_{\alpha}^{\beta} \rho \sin\theta \sqrt{\rho^2 + \rho'^2}\, d\theta \qquad (44.4''')$$

formula yordamida shu yoyning qutb o'qiga nisbatan inersiya momenti

$$I_\rho = I_x = \int\limits_{\alpha}^{\beta} \rho^2 \sin^2\theta \sqrt{\rho^2 + \rho'^2}\, d\theta \qquad (44.5''')$$

formula yordamida topiladi.

Endi (44.3') formuladan foydalanib Guldenning 1-teoremasini isbotlaymiz. Bu formulani

$$2\pi y_C \cdot \int\limits_a^b \sqrt{1 + y'^2}\, dx = 2\pi \int\limits_a^b y\sqrt{1 + y'^2}\, dx \text{ yoki } 2\pi y_C \cdot S = P_x$$

ko'rinishda yozamiz, bunda *S-AB* egri chiziq yoyining uzunligi, P_x-*AB* yoyni $0x$ o'q atrofida aylanishidan hosil bo'lgan sirt yuzi. Oxirgi tenglikdagi $2\pi y_C$ *AB* yoyni og'irlik markazini $0x$ o'q atrofida aylanishi natijasida chizgan aylananing uzunligi ekanini hisobga olsak $2\pi y_C \cdot S = P_x$ tenglik Gulden teoremasini to'g'riligini tasdiqlaydi.

1-misol. ox o'qning yuqorisida joylashgan $x^2+y^2=a^2$ yarim aylana og'irlik markazinining koordinatalari topilsin.

Yechish.

$$y=\sqrt{a^2-x^2},\quad y'=-\frac{x}{\sqrt{a^2-x^2}},\quad \sqrt{1+y'^2}=\sqrt{1+\frac{x^2}{a^2-x^2}}=\frac{a}{\sqrt{a^2-x^2}}$$

(44.3') formulaga binoan

$$y_c=\frac{\int\limits_{-a}^{a}\sqrt{a^2-x^2}\cdot\frac{a}{\sqrt{a^2-x^2}}dx}{a\int\limits_{-a}^{a}\frac{dx}{\sqrt{a^2-x^2}}}=\frac{ax\Big|_{-a}^{a}}{a\arcsin\frac{x}{a}\Big|_{-a}^{a}}=\frac{2a^2}{2a\arcsin 1}=\frac{a}{\frac{\pi}{2}}=\frac{2a}{\pi}.$$

Qaralayotgan yarim aylana oy o'qqa nisbatan simmetrik bo'lganligi uchun $x_c=0$ bo'ladi.

Demak, $C(o;\ \frac{2a}{\pi})$ nuqta berilgan yarim aylananing og'irlik markazidir.

2-misol. $x=a\cos^3t$, $y=a\sin^3t$ astroidaning birinchi kvadrant (chorak) da yotgan yoyining ox va oy o'qlarga nisbatan statik momentlarini va og'irlik markazini toping (162-chizma).

Yechish. $x'=-3a\cos^2t\sin t$, $y'=3a\sin^2t\cos t$,

$$\sqrt{x'^2+y'^2}=\sqrt{9a^2\cos^4t\sin^2t+9a^2\sin^4t\cos^2t}=$$

$$=\sqrt{9a^2\cos^2t\sin^2t(\cos^2t+\sin^2t)}=3a\cos t\sin t.$$

$$=3a^2\int\limits_0^{\frac{\pi}{2}}\sin^4t\,d(\sin t)=3a^2\frac{\sin^5t}{5}\Big|_0^{\frac{\pi}{2}}=\frac{3a^2}{5}.$$

(44.4") formulaga asosan:

$$M_x=\int_0^{\frac{\pi}{2}} a\sin^3 t\cdot 3a\cos t\sin t dt=3a^2\int_0^{\frac{\pi}{2}}\sin^4 t\cos t dt=$$

$$M_y=\int_0^{\frac{\pi}{2}} a\cos^3 t\cdot 3a\cos t\sin t dt=-3a^2\int_0^{\frac{\pi}{2}}\cos^4 t d\cos t=-\frac{3a^2}{5}\cos^5 t\Big|_0^{\frac{\pi}{2}}=\frac{3a^2}{5}$$

Massani topamiz:

$$M=\int_0^{\frac{\pi}{2}}\sqrt{x'^2+y'^2}dt=3a\int_0^{\frac{\pi}{2}}\cos t\sin t dt=3a\int_0^{\frac{\pi}{2}}\sin t d(\sin t)=3a\frac{\sin^2 t}{2}\Big|_0^{\frac{\pi}{2}}=\frac{3a}{2}.$$

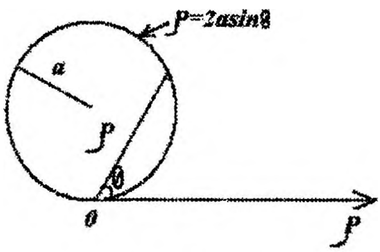
Demak,

$$x_c=\frac{M_y}{M}=\frac{3a^2}{5}:\frac{3a}{2}=\frac{2}{5}a=0{,}4a.$$

$$y_c=\frac{M_x}{M}=0{,}4a.$$

Shunday qilib $M_x=M_y=0{,}6a^2$, $C\ (0{,}4a;\ \ 0{,}4a)$.

3-misol. $\rho=2a\sin\theta$ aylananing qutb o‘qiga nisbatan statik momentini toping (193-chizma).



193-chizma.

Yechish. Qutb burchagi θ 0 dan π gacha o‘zgarganda aylana chiziladi. Shuning uchun (44.4‴) formulaga binoan quyidagiga ega bo'lamiz.

$$M_\rho = M_x = \int_0^\pi 2a\sin\theta \sin\theta\sqrt{(2a\sin\theta)^2 + (2a\cos\theta)^2}\,d\theta =$$

$$= 4a^2\int_0^\pi \sin^2\theta d\theta = 4a^2\int_0^\pi \frac{1-\cos 2\theta}{2}d\theta = 2a^2(\theta - \frac{1}{2}\sin 2\theta)\Big|_0^\pi = 2a^2\pi.$$

44.4. Yassi (tekis) figura og'irlik markazi va statik hamda inersiya momentlari

Berilgan figura uzluksiz $y=f_1(x)$, $y=f_2(x)$, $(f_1(x)\le f_2(x))$ egri chiziqlar hamda $x=a$, $x=b$ $(x<b)$ to'g'ri chiziqlar bilan chegaralangan bo'lsin. Uning sirt zichligi, ya'ni yuz birligiga mos massa hamma joyda bir xil va δ ga teng deb faraz qilamiz (δ =const). Bunaqa figura odatda bir jinsli deb yuritiladi. Shu figuraning og'irlik markazini topamiz. (194-chizma).

Berilgan figurani $x=a=x_o$, $x=x_1$, $x=x_2$,... $x=x_n=b$ to'g'ri chiziqlar bilan kengligi $\Delta x_1, \Delta x_2,...,\Delta x_n$ bo'lgan n ta bo'laklarga ajratamiz.

Har bir bo'lakning massasi uning yuzi bilan δ chiziqli zichlikning ko'paytmasiga teng bo'ladi.

Agar i-bo'lakni asosi Δx_i va balandligi $f_2(z_i)$-$f_1(z_i)$ bo'lgan to'g'ri to'rtburchak bilan almashtirsak, i-bo'lakning massasi

$$\Delta m_i = \delta[f_2(z_i) - f_1(z_i)]\,\Delta x_i$$

ga teng bo'ladi. bunda $z_i = \frac{x_{i-1} + x_i}{2}$ (ya'ni z_i $[x_{i-1}, x_i]$ kesmaning o'rtasi). Bu bo'lakning og'irlik markazi taxminan tegishli to'g'ri to'rtburchakning markazi(diogonallarining kesishish nuqtasi) da, ya'ni koordinatalari

$$(x_i)_c = z_i, \qquad (y_i)_c = \frac{f_2(z_i) + f_i(z_i)}{2}$$

bo'lgan nuqtada bo'ladi.

Endi har bir bo'lakni massasi tegishli to'g'ri to'rtburchakning massasiga teng bo'lgan va shu to'g'ri to'rtburchak og'irlik markaziga to'plangan moddiy nuqta bilan almashtiramiz.

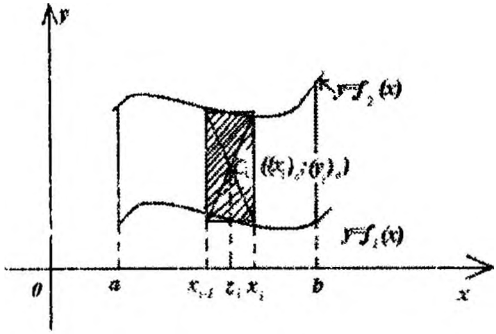
U holda butun figura og'irlik markazi koordinatalarini topish uchun moddiy nuqtalar sistemasi og'irlik markazi koordinatalarini topish formulalari (44.2) va (44.3) dan foydalanish mumkin. Bu formulalarga x_i o'rniga z_i, y_i o'rniga $\frac{1}{2}[f_2(z_i)+f_1(z_i)]$, m_i o'rniga $\Delta m_i = \delta[f_2(z_i)-f_1(z_i)]\Delta x_i$ ni quyib quyidagiga ega bo'lamiz.

$$x_c \approx \frac{\sum_{i=1}^{n} z_i \delta[f_2(z_i)-f_1(z_i)]\Delta x_i}{\sum_{i=1}^{n} \delta[f_2(z_i)-f_1(z_i)]\Delta x_i},$$

$$y_c \approx \frac{\frac{1}{2}\sum_{i=1}^{n}[f_2(z_i)+f_1(z_i)]\delta[f_2(z_i)-f_1(z_i)]\Delta x_i}{\sum_{i=1}^{n} \delta[f_2(z_i)-f_1(z_i)]\Delta x_i}$$

Bunda $\max\Delta x_i \to o$ da limitga o'tib, berilgan figura og'irlik markazining koordinatalarini topamiz:

$$x_c = \frac{\int_a^b x[f_2(x)-f_1(x)]dx}{\int_a^b [f_2(x)-f_1(x)]dx}, \quad y_c = \frac{\frac{1}{2}\int_a^b [f_2(x)+f_1(x)][f_2(x)-f_1(x)]dx}{\int_a^b [f_2(x)-f_1(x)]dx} \qquad (44.6)$$



194-chizma.

Bu formulalarning maxrajida berilgan figuraning massasi turishini hamda chiziqli zichlik δ o'zgarmas bo'lganligi tufayli hisoblash jarayonida qisqarib ketganligini qayd etamiz. Berilgan figura bir jinsli bo'lmasa, ya'ni chiziqli zichlik δ o'zgaruvchi bo'lganda figuraning og'irlik markazi koordinatalari chiziqli zichlikka bog'liq bo'ladi.

Shunday qilib (44.6) formulalardan chiziqli zichlik $\delta=1$ deb faraz qilib

$$M_x = \frac{1}{2}\int_a^b [f_2(x)+f_1(x)][f_2(x)-f_1(x)]dx, \qquad (44.7)$$

va

$$M_y = \int_a^b x[f_2(x)-f_1(x)]dx \qquad (44.8)$$

berilgan bir jinsli figuraning *ox* va *oy* o'qlarga nisbatan statik momentlarini topish formulalarini hamda

$$M = \int_a^b [f_2(x)-f_1(x)]dx \qquad (44.9)$$

shu figuraning masasini topish formulasini hosil qilamiz.

Agar berilgan bir jinsli figura (zichlik $\delta=1$) $y=f(x)\geq o$, $y=0$, $x=a$, $x=b$ $(a<b)$ chiziqlar bilan chegaralangan bo'lsa, u holda yuqorida

keltirilgan formulalar quyidagi ko'rinishni oladi:

$$M_x = \frac{1}{2}\int_a^b y^2 dx = \frac{1}{2}\int_a^b f^2(x)dx \quad (44.7'), \quad M_y = \int_a^b xydx = \int_a^b xf(x)dx \quad (44.8'),$$

$$M = \int_a^b ydx = \int_a^b f(x)dx. \qquad (44.9')$$

Shu figuraning koordinata o'qlariga nisbatan inersiya momentlari

$$I_x = \frac{1}{3}\int_a^b y^3 dx = \frac{1}{3}\int_a^b f^3(x)dx \quad (44.10), \quad I_y = \int_a^b x^2 ydx = \int_a^b x^2 f(x)dx \quad (44.11),$$

formulalar yordamida topilishini ta'kidlab o'tamiz.

(44.6) formuladan foydalanib Guldenning 2-teoremasini isbotlash qiyin emas.

Haqiqatdan (44.6) formulaning ikkinchi tengligini

$$2\pi y_C \cdot \int_a^b [f_2(x) - f_1(x)]dx = \pi \int_a^b [f_2^2(x) - f_1^2(x)]dx$$

ko'rinishda yozamiz. Bu tenglikning chap tomonidagi integral tekis figuraning yuzini, o'ng tomonidagi integral shu figurani $0x$ o'q atrofida aylanishidan hosil bo'lgan jismning hajmi V_x ni ifodalashini hisobga olsak

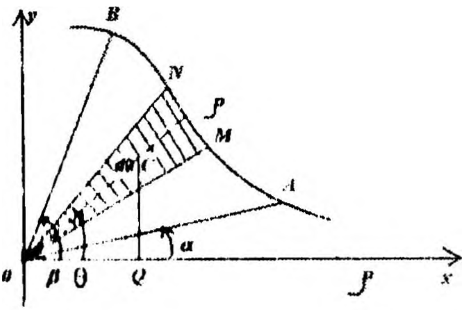
$$2\pi y_C \cdot Q = V_x$$

tenglikka ega bo'lamiz, bunda $2\pi y_C$ tekis figura og'irlik markazi $C(x_C; y_C)$ nuqtaning $0x$ o'q atrofida aylanishi natijasida chizgan aylananing uzunligi. Oxirgi tenglik Gulden teoremasining to'g'riligini tasdiqlaydi.

Endi ikkita $\theta = \alpha$, $\theta = \beta$ $(\alpha < \beta)$ nurlar va qutb koordinatalar sistemasida $\rho = \rho(\theta)$ tenglamaga ega uzluksiz egri chiziq bilan chegaralangan bir jinsli sektorning og'irlik markazining abssissasi va ordinatasi (195-chizma)

$$x_c = \frac{2}{3}\frac{\int\limits_{\alpha}^{\beta}\rho^3\cos\theta d\theta}{\int\limits_{\alpha}^{\beta}\rho^2 d\theta}, \qquad y_c = \frac{2}{3}\frac{\int\limits_{\alpha}^{\beta}\rho^3\sin\theta d\theta}{\int\limits_{\alpha}^{\beta}\rho^2 d\theta}, \qquad (44.12)$$

formulalar orqali topilishini isbotlaymiz



195-chizma.

Egri chiziqli OMN sektorni qaraylik. Kichik $d\theta$ lar uchun bu sektorning yuzi

$$S_{OMN} = \frac{1}{2}\rho^2 d\theta$$

bo'ladi. Uchburchakning og'irlik markazi uning medianalari kesishgan nuqtada joylashganligi elementar geometriyadan ma'lum. Egri chiziqli OMN sektorni og'irlik markazi C nuqtada va massasi $\Delta m = \frac{1}{2}\rho^2 d\theta$ uning og'irlik markazida joylashgan uchburchak deb faraz qilib, uning $0x$ o'qqa nisbatan statik momentini hisoblaymiz. Δ OCQ dan

$$\frac{CQ}{OC} = \sin\theta, \qquad CQ = OC\sin\theta.$$

Uchburchak medianasining xossasiga ko'ra mediana kesishish nuqtasida uning uchidan hisoblanganda *2:1* nisbatda bo‘linadi. Shunga ko‘ra $OC=\frac{2}{3}\rho$ va $CQ=\frac{2}{3}\rho\sin\theta$ bo‘ladi.

Demak, *OMN* sektorning *0x* o‘qqa nisbatan statik momenti

$$dM_x = CQ\cdot dm = \frac{2}{3}\rho\sin\theta\cdot\frac{1}{2}\rho^2 d\theta = \frac{1}{3}\rho^3\sin\theta d\theta$$

bo‘ladi. Bu tenglikni α dan β gacha integrallab *OAB* sektorning *0x* o‘qqa (qutb o‘qiga) nisbatan statik momentini hosil qilamiz:

$$M_x = \frac{1}{3}\int_\alpha^\beta \rho^3\sin\theta d\theta\,. \tag{44.13}$$

Xuddi shunday

$$M_y = \frac{1}{3}\int_\alpha^\beta \rho^3\cos\theta d\theta \tag{44.14}$$

ekanini topamiz.

Ma‘lumki, *OAB* sektorning yuzi

$$S = \frac{1}{2}\int_\alpha^\beta \rho^2 d\theta$$

formula yordamida topiladi. Figuraning yuzi uning massasiga teng ekanligini nazarda to‘tib (geometrik figuralarning chiziqli zichligi birga teng deb olinadi)

$$x_c = \frac{M_y}{M} = \frac{M_y}{S},\quad y_c = \frac{M_x}{M} = \frac{M_x}{S}$$

tengliklarga M_y, M_x va S o‘rniga ularning topilgan qiymatlarini qo‘ysak isbotlanishi lozim bo‘lgan (44.12)formulalar hosil bo‘ladi.

4-misol. $y=\sin x$ sinusoida yoyi va *0x* o‘q bilan chegaralangan figura ($o\le x\le\pi$) og‘irlik markazini toping.

Yechish. Berilgan figura $x=\frac{\pi}{2}$ to‘g‘ri chiziqqa nisbatan simmetrik bo'lganligi sababli $x_c = \frac{\pi}{2}$ bo'ladi. (44.7′) formulaga binoan:

$$M_x = \frac{1}{2}\int_0^{\pi} \sin^2 x dx = \frac{1}{2}\int_0^{\pi} \frac{1-\cos 2x}{2} dx = \frac{1}{4}\int_0^{\pi} (1-\cos 2x)dx =$$

$$\frac{1}{4}(x - \frac{1}{2}\sin 2x)\Big|_o^{\pi} = \frac{1}{4}\cdot\pi = \frac{\pi}{4}.$$

(44.9′) formulaga binoan figuraning massasi

$$M = \int_0^{\pi} \sin x dx = -\cos x\Big|_o^{\pi} = 2$$

bo'ladi Demak,

$$y_c = \frac{M_x}{M} = \frac{\pi}{4} : 2 = \frac{\pi}{8}$$

va figuraning og'irlik markazi $C(\frac{\pi}{2};\frac{\pi}{8})$ ekan.

5-misol. $\begin{cases} x = a\cos^3 t, \\ y = a\sin^3 t. \end{cases}$ $o \le t \le 2\pi$ astroidaning birinchi kvadranti (chorak) dagi qismi va koordinata o'qlari bilan chegaralangan figuraning og'irlik markazi topilsin (162-chizma).

Yechish. Simmetriyaga ko'ra $x_c = y_c$. Statik moment M_y ni (44.8′) formuladan foydalanib topamiz.

$$M_y = \int_o^a xydx = \int_{\frac{\pi}{2}}^0 a\cos^3 ta \cdot \sin^3 t(a\cos^3 t)' dt =$$

$$= 3a^3 \int_{\frac{\pi}{2}}^0 \cos^3 t \cdot \sin^3 t \cos^2 t \sin t dt = 3a^3 \int_0^{\frac{\pi}{2}} \cos^5 t \cdot \sin^4 t dt =$$

$$=3a^3\int_0^{\frac{\pi}{2}}\cos^4 t\cdot\sin^4 t\cos t dt=3a^3\int_0^{\frac{\pi}{2}}(1-\sin^2 t)^2\cdot\sin^4 t d\sin t=$$

$$=3a^3\int_0^{\frac{\pi}{2}}(\sin^4 t-2\sin^6 t+\sin^8 t)d\sin t=$$

$$=3a^3\left(\frac{\sin^5 t}{5}-2\cdot\frac{\sin^7 t}{7}+\frac{\sin^9 t}{9}\right)\Bigg|_0^{\frac{\pi}{2}}=3a^3\left(\frac{1}{5}-\frac{2}{7}+\frac{1}{9}\right)=\frac{8a^3}{105}.$$

Figura massasi (44.9′) ga ko‘ra

$$M=S=\int_a^a ydx=\int_{\frac{\pi}{2}}^0 a\sin^3 t(a\cos^3 t)'dt=-a^2\int_{\frac{\pi}{2}}^0 \sin^3 t3\cos^2 t\sin t dt=3a^2\int_0^{\frac{\pi}{2}}\sin^4 t\cos^2 t dt=$$

$$=3a^2\int_0^{\frac{\pi}{2}}(\sin^2 t)^2\cos^2 t dt=3a^2\int_0^{\frac{\pi}{2}}\left(\frac{1-\cos 2t}{2}\right)^2\cdot\frac{1+\cos 2t}{2}dt=\frac{3a^2}{8}\int_0^{\frac{\pi}{2}}(1-\cos 2t)(1-\cos^2 2t)dt=$$

$$=\frac{3a^2}{8}\int_0^{\frac{\pi}{2}}(1-\cos 2t)\sin^2 2t dt=\frac{3a^2}{8}\int_0^{\frac{\pi}{2}}(\sin^2 2t-\sin^2 2t\cdot\cos 2t)dt=$$

$$=\frac{3a^2}{8}\int_0^{\frac{\pi}{2}}\left(\frac{1-\cos 4t}{2}-\sin^2 2t\cdot\cos 2t\right)dt=$$

$$=\frac{3a^2}{16}\int_0^{\frac{\pi}{2}}(1-\cos 4t)dt-\frac{3a^2}{8}\int_0^{\frac{\pi}{2}}\sin^2 2t\cdot\frac{1}{2}d(\sin 2t)=\frac{3a^2}{16}(t-\frac{1}{4}\sin 4t)\Big|_0^{\frac{\pi}{2}}-\frac{3a^2}{16}\cdot\frac{\sin^3 2t}{3}\Big|_0^{\frac{\pi}{2}}=\frac{3a^2\pi}{32}$$

ga teng ekanligi kelib chiqadi.

U holda $\quad x_c=y_c=\frac{M_y}{M}=\frac{8a^3\cdot 32}{105\cdot 3\pi a^2}=\frac{256a}{315\pi}.$

Demak, figuraning og‘irlik markazi $C=\left(\frac{256a}{315\pi},\frac{256a}{315\pi}\right)$ ekan.

6-misol. $\rho=a(1+\cos\theta)$ kardioida bilan chearalangan figura og‘irlik markazining dekart koordinatalarini toping (166-chizma).

Yechish. Figuraning $0x$ o‘qqa simmetrikligidan $y_c=0$ ekanligi kelib chiqadi.

(44.12)formulalarning birinchisiga ko'ra:

$$x_c = \frac{2}{3}\cdot\frac{a^3\int\limits_0^{2\pi}(1+\cos\theta)^3\cos\theta d\theta}{a^2\int\limits_0^{2\pi}(1+\cos\theta)^2 d\theta}.$$

Suratdagi integralni hisoblaymiz:

$$\int\limits_0^{2\pi}(1+\cos\theta)^3\cos\theta d\theta=\int\limits_0^{2\pi}(1+3\cos\theta+3\cos^2\theta+\cos^3\theta)\cos\theta d\theta=$$

$$\int\limits_0^{2\pi}(\cos\theta+3\cos^2\theta+3\cos^3\theta+\cos^4\theta)d\theta=$$

$$\int\limits_0^{2\pi}\cos\theta d\theta+3\int\limits_0^{2\pi}\frac{1+\cos 2\theta}{2}d\theta+3\int\limits_0^{2\pi}\cos^2\theta\cos\theta d\theta+\int\limits_0^{2\pi}\left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta=\sin\theta\Big|_0^{2\pi}+$$

$$+\frac{3}{2}(\theta+\frac{1}{2}\sin 2\theta)\Big|_0^{2\pi}+3\int\limits_0^{2\pi}(1-\sin^2\theta)d\sin\theta+\frac{1}{4}\int\limits_0^{2\pi}(1+2\cos 2\theta+\cos^2 2\theta)d\theta=$$

$$3\pi+3(\sin\theta-\frac{\sin^3\theta}{3})\Big|_0^{2\pi}+\frac{1}{4}(\theta+2\cdot\frac{1}{2}\sin 2\theta)\Big|_0^{2\pi}+\frac{1}{4}\int\limits_0^{2\pi}\cos^2 2\theta d\theta=3\pi+\frac{\pi}{2}+$$

$$+\frac{1}{8}\int\limits_0^{2\pi}(1+\cos 4\theta)d\theta=\frac{7\pi}{2}+\frac{1}{8}(\theta+\frac{1}{4}\sin 4\theta)\Big|_0^{2\pi}=\frac{7\pi}{2}+\frac{\pi}{4}=\frac{15\pi}{4}.$$

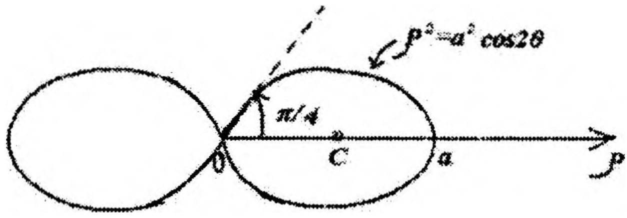
x_c ning maxrajidagi integral kardioida bilan chegaralangan figuraning yuzini ikkilangani bo'lgani uchun u $3a^2\pi$ ga teng (41-mavzudagi 8-misolga qarang).

Demak,

$$x_c=\frac{2}{3}\cdot\frac{a^3\cdot\frac{15\pi}{4}}{3a^2\pi}=\frac{5a}{6}.$$

Shunday qilib berilgan figuraning og'irlik markazi $C(\frac{5a}{6},0)$ bo'lar ekan.

7-misol. $\rho^2=a^2\cos 2\theta$ Bernulli lemniskatasining o'ng sirtmogi bilan chegaralangan figura og'irlik markazining dekart koordinatalarini toping (196-chizma).



196-chizma.

Yechish. $\theta=0$ bo'lganda qutb radiusi ρ eng katta qiymati a ga teng bo'ladi. $\theta=\frac{\pi}{4}$ da $\rho=0$. θ qutb burchak $-\frac{\pi}{4}$ dan $\frac{\pi}{4}$ gacha o'zgarganda $M(\theta,\rho)$ nuqta lemniskataning o'ng sirtmogini chizadi. Qutb burchak θ $\frac{3\pi}{4}$ dan $\frac{5\pi}{4}$ gacha o'zgarganda $M(\theta,\rho)$ nuqta harakatlanib lemniskataning chap sirtmogini chizadi. Figura ox (ρ) o'qqa nisbatan simmetrik joylashganligi sababli $y_c=0$.

$\rho^3=a^3(\cos 2\theta)^{\frac{3}{2}}$ ga egamiz. Integrallash chegaralari $\alpha=-\frac{\pi}{4}$ va $\beta=\frac{\pi}{4}$.

(44.12) formulaga binoan:

$$x_c=\frac{2}{3}\cdot\frac{\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}a^3(\cos 2\theta)^{\frac{3}{2}}\cos\theta d\theta}{\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}a^2\cos 2\theta d\theta}=\frac{2}{3}\cdot\frac{a\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(1-2\sin^2\theta)^{\frac{3}{2}}\cos\theta d\theta}{\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\cos 2\theta d\theta}=\frac{2}{3}a\cdot\frac{\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(1-2\sin^2\theta)^{\frac{3}{2}}\cos\theta d\theta}{\frac{1}{2}\sin 2\theta\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}}=$$

$$=\frac{2}{3}a\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(1-2\sin^2\theta)^{\frac{3}{2}}d(\sin\theta)\left|\begin{matrix}\sin\theta=\frac{1}{\sqrt{2}}\sin t,\cos\theta d\theta=\frac{1}{\sqrt{2}}\cos tdt\\ \theta=-\frac{\pi}{4}\,d\grave{a}\quad t=-\frac{\pi}{2},\theta=-\frac{\pi}{4}\,d\grave{a}\quad t=\frac{\pi}{2}\end{matrix}\right|=$$

$$=\frac{2}{3}a\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1-\sin^2 t)^{\frac{3}{2}}\cdot\frac{1}{\sqrt{2}}\cos tdt=\frac{2}{3\sqrt{2}}a\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos^3 t\cos tdt=$$

$$=\frac{2a}{3\sqrt{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(\cos^2 t)^2dt=\frac{2a}{3\sqrt{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left(\frac{1+\cos 2t}{2}\right)^2dt=\frac{2a}{3\sqrt{2}}\cdot\frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+2\cos 2t+\cos^2 2t)dt=$$

$$=\frac{a}{6\sqrt{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(1+2\cos 2t+\frac{1}{2}(1+\cos 4t))dt=\frac{a}{6\sqrt{2}}(t+2\cdot\frac{1}{2}\sin 2t+\frac{1}{2}t+\frac{1}{2}\cdot\frac{1}{4}\sin 4t)\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}=$$

$$=\frac{a}{6\sqrt{2}}(\pi+\frac{\pi}{2})=\frac{\pi a}{4\sqrt{2}}=\frac{\pi a\sqrt{2}}{8}.$$

Shunday qilib Bernulli lemniskatasining o‘ng sirtmogi bilan chegaralangan figuraning og‘irlik markazini dekart koordinatalari $x_c=\frac{\pi a\sqrt{2}}{8}$, $y_c=0$ ekan.

8-misol. Yarim o‘qlari a va b bo‘lgan ellipsning uning har ikkala o‘qiga nisbatan inersiya momentini toping.

Yechish. Ellipsning $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ tenglamasidan $y=\pm\frac{b}{a}\sqrt{a^2-x^2}$ ni topamiz. Ellips koordinata o‘qlariga nisbatan simmetrik bo‘lganligidan ellips bilan chegaralangan figura yuzining to‘rtdan birining inersiya momentini hisoblab, natijani *4* ga ko‘paytirish kifoya.

$y=\frac{b}{a}\sqrt{a^2-x^2}$, $x=0$, $y=0$ $(0\leq x\leq a)$ chiziqlar bilan chegaralangan figuraning $0x$ o‘qqa nisbatan inersiya momentini topamiz. (44.10) formulaga binoan:

$$\frac{1}{4}\cdot I_x=\frac{1}{3}\int_a^b y^3dx=\frac{1}{3}\int_0^a\left(\frac{b}{a}\sqrt{a^2-x^2}\right)^3dx=\frac{b^3}{3a^3}\int_0^a\left(\sqrt{a^2-x^2}\right)^3dx\left|\begin{array}{l}x=a\sin t\\ dx=a\cos tdt\\ x=0\ da\ t=0,\\ x=a\ da\ t=\frac{\pi}{2}\end{array}\right|=$$

$$=\frac{b^3}{3a^3}\int_0^{\frac{\pi}{2}}\left(\sqrt{a^2-a^2\sin^2 t}\right)^3a\cos tdt=\frac{b^3}{3a^3}\int_0^{\frac{\pi}{2}}(a\cos t)^3a\cos tdt=$$

$$=\frac{b^3a}{3}\int_0^{\frac{\pi}{2}}\cos^4 tdt=\frac{b^3a}{3}\int_0^{\frac{\pi}{2}}\left(\frac{1+\cos 2t}{2}\right)^2dt=\frac{b^3a}{12}\int_0^{\frac{\pi}{2}}\left(1+2\cos 2t+\cos^2 2t\right)dt=$$

$$=\frac{b^3a}{12}\int_0^{\frac{\pi}{2}}\left(1+2\cos 2t+\frac{1+\cos 4t}{2}\right)dt=\frac{b^3a}{12}\left(t+2\cdot\frac{1}{2}\sin 2t+\frac{1}{2}t+\frac{1}{2}\cdot\frac{1}{4}\sin 4t\right)\Bigg|_0^{\frac{\pi}{2}}=$$

$$=\frac{b^3a}{12}\left(\frac{\pi}{2}+\frac{\pi}{4}\right)=\frac{b^3a}{12}\cdot\frac{3\pi}{4}=\frac{b^3a\pi}{16}.$$

Demak, $I_x=\frac{b^3a\pi}{16}\cdot 4=\frac{a\pi b^3}{4}.$

Xuddi Shunga o‘xshash (44.11) formuladan foydalanib $I_y=\frac{\pi a^3b}{4}$. ni ham topish mumkin.

Olingan natijalardan $a=b$ bo‘lganda a radiusli bir jinsli $(\delta=1)$ doiraning uning diametriga nisbatan inersiya momentini hosil qilamiz:

$$I=\frac{\pi a^4}{4}.$$

9-misol. R radiusli bir jinsli (zichlik $\delta=1$) doiraning qutb inersiya momentini, ya'ni doiraning uning markaziga nisbatan inersiya momentini toping.

Yechish. m massali M moddiy nuqtaning P nuqtaga nisbatan inersiya momenti deb $I_o=md^2$ songa aytiladi, bu yerdagi d M va P nuqtalar orasidagi masofa. Doira markazidan ρ masofada yotuvchi $d\rho$ qalinlikdagi (kenglikdagi) doiraviy halqani olamiz, bunda $d\rho$ istalgancha kichik miqdor. Shartga ko'ra halqa bir jinsli bo'lganligi sababli uning massasi halqaning yuzi $\pi(\rho+d\rho)^2 - \pi\rho^2 \approx 2\pi\rho\, d\rho$ ga teng. Demak halqaning doira markaziga nisbatan inersiya momenti:

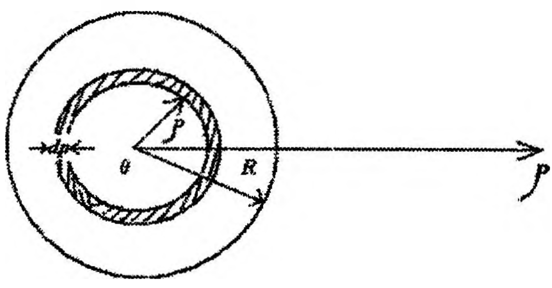
$$dI_o \approx \rho^2 2\pi\rho\, d\rho = 2\pi\rho^3\, d\rho$$

va doiraning uning markaziga nisbatan inersiya momenti:

$$I_0 = 2\pi\int_0^R \rho^3 d\rho = 2\pi\cdot\frac{\rho^4}{4}\Big|_0^R = \frac{\pi R^4}{2} = \frac{MR^2}{2},$$

bu yerda $M=\pi R^2$ -doira massasi (yuzi).

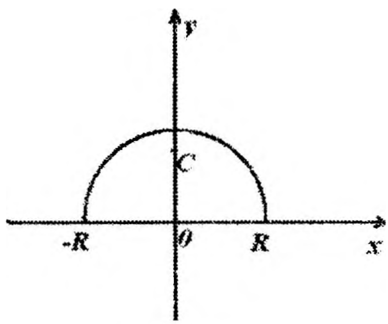
Biz yuqorida keltirgan misollarimizda qaralayotgan geometrik figuralar bir jinsli, ya'ni chiziqli zichlik birga teng deb faraz qildik.



197-chizma.

10-misol. Guldenning birinchi teoremasidan foydalanib radiusi R ga teng yarim aylananing og'irlik markazi topilsin.

Yechish. Koordinatalar sistemasini 198-chizmada ko'rsatilganidek tanlaymiz. Yarim aylana Oy ga simmetrik bo'lgani uchun $x_C = 0$. Yarim aylanani Ox atrofida aylanishidan hosil bo'lgan sferaning yuzi $P_x = 4\pi R^2$, yarim aylananing uzunligi πR bo'lgani uchun Guldenning birinchi teoremasiga binoan $4\pi R^2 = \pi R \cdot 2\pi y_C$, bundan $y_C = 2\frac{R}{\pi}$ bo'ladi.



198-chizma.

Demak, yarim aylananing og'irlik markazi $C(0;\frac{2R}{\pi})$ nuqtada bo'lar ekan.

11-misol. Guldenning ikkinchi teoremasidan foydalanib

$$x = a(t - \sin t),\ y = a(1 - \cos t)$$

sikloidaning bitta arkasi hamda Ox o'q bilan chegaralangan figuraning og'irlik markazi topilsin (173-chizma).

Yechish. Figura $x = \pi a$ to'g'ri chiziqqa nisbatan simmetrik bo'gani uchun uning og'irlik markazi shu tog'ri chiziqda yotadi, ya'ni $x_C = \pi a$.

Figurani Ox o'q atrofida aylanishidan hosil bo'lgan jismning hajmi $V_x = 5\pi^2 a^3$ (43.7-misol) va figurani yuzi $S = 3\pi a^2$ ekanini hisobga olsak Guldenning ikkinchi teoremasiga ko'ra

$$y_C = \frac{V_x}{2\pi S} = \frac{5\pi^2 a^3}{2\pi \cdot 3\pi a^2} = \frac{5a}{6}$$

bo'ladi. Demak, $C(\pi a; \frac{5a}{6})$ nuqta qaralayotgan figuraning og'irlik markazi bo'lar ekan.

O'z-o'zini tekshirish uchun savollar

1. O'qqa nisbatan statik moment nima?
2. Nuqtaga va o'qqa nisbatan inersiya momenti nima?
3. Moddiy nuqtalar sistemasining og'irlik markazi koordinatalarini topish formulalarini yozing.
4. Moddiy nuqtalar sistemasining koordinata o'qlariga nisbatan statik momentlarini topish formulalarini yozing.
5. Moddiy nuqtalar sistemasining koordinatalar boshiga nisbatan inersiya momentini topish formulasini yozing.
6. Moddiy nuqtalar sistemasining koordinata o'qlariga nisbatan inersiya momentalrini topish formulalarini yozing.
7. Tekislikda chiziq $y=f(x)$ tenglama bilan va parametrik tenglamalar hamda u qutb koordinatalar sistemasida berilaganda uning koordinata o'qlariga nisbatan statik va inersiya momentlari hamda og'irlik markazi qanday topiladi?
8. $y=f_1(x)$, $y=f_2(x)$, $x=a$, $x=b$ $(y=f_1(x) \leq y=f_2(x)$, $a<b)$ chiziqlar bilan chigaralangan tekis figura og'irlik markazi hamda uning o'qlarga nisbatan statik va inersiya momentlari qanday topiladi?
9. $\theta=\alpha$, $\theta=\beta$ $(\alpha<\beta)$ nurlar va qutb koordinatalar sistemasida $\rho=\rho(\theta)$ tenglamaga ega egri chiziq bilan chegaralangan egri chiziqli sektorning og'irlik markazini abssissasi va ordinatasini hamda koordinata o'qlari $0x$, $0y$ larga nisbatan statik momentini topish formulalarini yozing.

10. $y=f(x)$, $y=0$, $x=a$, $x=b$ chiziqlar bilan chegaralangan bir jinsli figura og'irlik markazining koordinatalari hamda uning koordinata o'qlariga nisbatan statik va inersiya momentlari qanday topiladi?

Mustaqil yechish uchun mashqlar

1. $y=2\sqrt{x}$ egri chiziq yoyining $x=3$ to'g'ri chiziq kesgan qismining $0x$ o'qqa nisbatan statik momentini toping.

Javob: $M_x=\frac{28}{3}$.

2. $x=a(t-\sin t)$, $y=a(1-\cos t)$ sikloida birinchi arkasining og'irlik markazi koordinatalarini toping.

Javob: $C(\pi a;\frac{4a}{3})$

3. $\rho=4\cos\theta$ aylananing qutbdan qutb o'qiga perpendikulyar bo'lib o'tgan to'g'ri chiziqqa nisbatan statik momentini toping.

Javob: 8π.

4. $x^2+4y-16=0$ parobola va $0x$ o'q bilan chegaralangan bir jinsli figuraning og'irlik markazi topilsin.

Javob: $C(0;\ 8/5)$.

5. $\rho=2(1-\cos\theta)$ kardioida bilan chegaralangan figura og'irlik markazining dekart koordinatalarini toping.

Javob: $C(\frac{5}{3};0)$

6. $y=4\sqrt{x}$ parabola yoyini $x=4$ to'g'ri chiziq kesgan qismining abssissalar o'qiga nisbatan inersiya momentini toping.

Javob: $32(6\sqrt{2}-\ln(3+2\sqrt{2}))$.

7. $y=2-x^2$ va $y=x^2$ chiziqlar bilan chegaralangan figuraning koordinata o'qlariga nisbatan inersiya momentini toping.

Javob: $I_y=\frac{8}{15}$, $I_x=\frac{356}{105}$.

45. ANIQ INTEGRALNING FIZIKA MASALALARINI YECHISHGA TATBIQLARI

45.1. O'zgaruvchan tezlikka ega nuqtaning bosib o'tgan yo'li

Aniq integralning geometriya va mexanikaga tatbiqlarida ko'rdikki u yoki bu masalalarni yechish uchun berilgan geometrik figura n ta ixtiyoriy qismlarga ajratilib qo'yilgan masala avval figuraning bitta qismi (elementar bo'lagi) uchun hal etiladi. Keyin olingan natijani jamlab integral yig'indi tuziladi. Integral yig'indida limitga o'tilsa qo'yilgan masalani yechish uchun aniq formula chiqarildi.

Bundan buyon fizika masalalarini yechishda ham shu usuldan foydalanamiz.

Faraz qilaylik nuqta o'zgaruvchan v tezlik bilan to'g'ri chiziqli harakat qilayotgan bo'lsin. v tezlik t vaqtning ma'lum funksiyasi, ya'ni $v=f(t)$ bo'lsin. Nuqtani vaqtning t_0 paytidan T paytigacha bosib o'tgan yo'lini aniqlash talab etilsin. $[t_0, T]$ oraliqni n ta ixtiyoriy $[t_0, t_1], [t_1, t_2], \dots, [t_{i-1}, t_i], \dots, [t_{n-1}, t_n]$ $(t_n=T)$ qismlarga ajratamiz. Har bir $[t_{i-1}, t_i]$ $(i=\overline{1,n})$ bo'lakda ixtiyoriy z_i nuqta olib bu oraliqda nuqtaning tezligi o'zgarmas va u $f(z_i)$ ga teng deb faraz qilamiz. U holda nuqtaning $\Delta t_i = t_i - t_{i-1}$ vaqt oralig'ida bosib o'tgan yo'li ΔS_i taqriban $f(z_i)\,\Delta t_i$ ga teng bo'lishi ayon. Nuqtaning butun $[t_0, T]$ oraliqda o'tgan yo'li

$$S \approx \sum_{i=1}^{n} f(z_i)\Delta t_i$$

bo'ladi. Bu yig'indi $[t_0, T]$ kesmada $f(t)$ funksiya uchun integral yig'indi ekanligini hisobga olib oxirgi taqribiy tenglikda $\max \Delta t_i \to 0$ da limitga o'tsak

$$S = \int_{t_0}^{T} v(t)dt = \int_{t_0}^{T} f(t)dt \qquad (45.1)$$

kelib chiqadi. Bu $[t_0,T]$ vaqt oralig'ida nuqtaning bosib o'tgan yo'li-ni topish formulasidir.

Eslatma. $[t_0,T]$ oraliqda o'zgarmas v tezlik bilan to'g'ri chiziqli harakat qilayotgan nuqtaning shu vaqt oralig'ida bosib o'tgan yo'li

$$S=v(T-t_0)$$

kabi aniqlanishi maktab kursidan ma'lum.

1-misol. Nuqtaning tezligi $v=(3t^2+2t+1)$ *m/s* ga teng. Harakat boshlangandan so'ng o'tgan $t=10$ *s* ichida nuqta bosib o'tgan S yo'lini toping.

Yechish. Shartga ko'ra $f(t)=3t^2+2t+1$, $t_0=0$, $T=10$. (45.1) formulaga binoan

$$S=\int_0^{10}(3t^2+2t+1)dt=(t^3+t^2+t)\Big|_0^{10}=1110(m).$$

2-misol. Nuqtaning tezligi $v=(9t^2-8t)$ *m/s* ga teng. Nuqtaning 4-sekundda bosib o'tgan S yo'lini toping.

Yechish. Shartga ko'ra $f(t)=9t^2-8t$, $t_0=3$, $T=4$.

Demak,

$$S=\int_3^4(9t^2-8t)dt=\left(9\cdot\frac{t^3}{3}-8\cdot\frac{t^2}{2}\right)\Big|_3^4=(3t^3-4t^2)\Big|_3^4=83(m).$$

3-misol. Nuqtaning tezligi $v=(12t-3t^2)$ *m/s* ga teng. Nuqtaning harakat boshlanganidan uning to'xtagunicha o'tgan vaqt oralig'ida bosib o'tgan S yo'li topilsin.

Yechish. Nuqtaning tezligi harakat boshlangunicha va nuqta to'xtaganida nolga tengligini hisobga olib $12t-3t^2=0$ yoki $t(4-t)=0$ tenglamani yechib $t_0=0, t=4$ ekanini topamiz.

Demak,

$$S=\int_0^4(12t-3t^2)dt=(6t^2-t^3)\Big|_0^4=32(m)$$

4-misol. Jism yer sathidan yuqoriga vertikal yo'nalishda $v=(39,2-9,8t)$ *m/s* tezlik bilan otildi. Shu jism yerdan qancha balandlikka ko'tarilishini toping.

Yechish. Jism yerdan eng yuqoriga ko'tarilganda uning tezligi $v=0$ ya'ni $39,2-9,8t=0$, bundan $t=4s$ kelib chiqadi. Shuning uchun (45,1) formulaga binoan

$$S=\int_0^4(39,2-9,8t)dt=(39,2t-4,9t^2)\Big|_0^4=78,4(m)$$

kelib chiqadi.

5 misol. *48km/soat* tezlik bilan harakatlanayotgan avtomobil tormoz berib tezligini kamaytira boshladi va *3* sek. dan keyin to'xtadi. Avtomobil butunlay to'xtaguncha qancha masofani bosib o'tishini toping (ishqalanishni va havoning qarshiligini hisobga olmang).

Yechish. Tekis sekinlanuvchan harakatning tezligi

$$v=v_0-at$$

formula orqali topiladi, bu yerda v_0-boshlang'ich tezlik, a-tezlanish.

Masalaning shartiga ko'ra

$$v_0=48\ km/soat=48\cdot\frac{1000m}{3600s}=\frac{40}{3}\cdot\frac{m}{s}.$$

Tezlanish a ni avtomobil *3 sek.* dan keyin to'xtash shartidan, ya'ni $t=3$ sekunda $v=0$ shartdan topamiz:

$$0=\frac{40}{3}-a\cdot 3,\qquad a=\frac{40}{9}\frac{m}{sek^2}.$$

v_0 va a ning qiymatini tezlikning formulasiga qo'yib, topamiz:

$$v=\frac{40}{3}-\frac{40}{9}t.$$

Demak,

$$S=\int_0^3\left(\frac{40}{3}-\frac{40}{9}t\right)dt=\left(\frac{40}{3}t-\frac{40}{9}\cdot\frac{t^2}{2}\right)\Big|_0^3=40-20=20\ \ (m).$$

6 misol. Reaktiv samolyot *20* sekund ichida o'z tezligini *360 km/soat* dan *720 km/soat* ga oshirdi. Samolyotning tezligini tekis tezlanuvchan deb hisoblab, u qanday tezlanish bilan uchganini va shu vaqt oralig'ida qancha masofani bosib o'tganini toping.

Yechish. Tekis tezlanuvchan harakat tezligi

$$v=v_0+at$$

formula orqali ifodalanadi. Masalaning shartiga ko'ra

$$t=0 \text{ da } \quad v_0 = 360 \ km/soat = 360 \cdot \frac{1000}{3600}\frac{m}{s} = 100\frac{m}{s},$$

$$t=20\ c \text{ da } \quad v = 720 \ km/soat = 720 \cdot \frac{1000}{3600}\frac{m}{s} = 200\frac{m}{s}.$$

v_0 va t ning qiymatlarini tezlikning formulasiga qo'yib, a tezlanishni topamiz:

$$200=100+a\cdot 20, \qquad a=5m/sek^2.$$

Demak, samolyotning tezligi

$$v=(100+5t)m/s$$

bo'ladi. (45.1) formulaga ko'ra

$$S = \int_0^{20}(100-5t)dt = (100t - 5\cdot\frac{t^2}{2})\Big|_0^{20} = 3000(m) = 3 \ km.$$

7-masala. *294m* balandlikdan pastga vertikal yo'nalishda *19,6 m/s* boshlang'ich tezlik bilan jism tushadi. Necha sekunddan keyin jism yerga kelib tushadi? (Og'irlik kuchi tezlanishi $g=9,8 \ m/sek^2$).

Yechish. Erkin tushayotgan jism tezligi (havoning qarshiligini hisobga olmaganda)

$$v=v_0+gt$$

formula orqali ifodalanadi. $v=19,6+9,8t$ ga egamiz.

Jismning tushish vaqti x ni

$$h = \int_0^{x}(19,6+9,8t)dt = (19,6t + 9,8\cdot\frac{t^2}{2})\Big|_0^{x} = 19,6x + 4,9x^2$$

tenglamadan topamiz, bu yerda $h=294 \ m$.

$$4,9x^2+19,6x-294=0$$

tenglamani yechamiz. Uni *4,9* ga qisqartirsak

$$x^2+4x-60=0$$

tenglama hosil bo'ladi. Bu tenglama

$x_{1,2} = -2 \pm \sqrt{4+60} = -2 \pm 8$ ya'ni $x_1=-10$, $x_2=6$ ildizlarga ega. Shartga ko'ra $x>0$ ekanini hisobga olsak faqat $x=6$ masalaning yechimi bo'lishini ko'ramiz. Shunday qilib jism $t=6$ *sek* dan keyin yerga kelib tushar ekan.

45.2 O'zgaruvchan kuchning bajargan ishi

M moddiy nuqta *F* kuch ta'siri ostida *0x* to'g'ri chiziq bo'ylab harakatlanayotgan bo'lsin va bunda kuchning yo'nalishi harakat yo'nalishi bilan bir xil bo'lsin (*F* kuch *0x* o'qqa parallel va ular bir xil yo'nalgan).

Shu *F* kuchning *M* moddiy nuqtani $x=a$ vaziyatdan $x=b$ vaziyatga ko'chirishda bajarogan ishi *A* ni topish talab etilsin.

Bunda ikki holatni kuzatish mumkin.

1. *F* kuch o'zgarmas bo'lsin. U holda nuqtani $x=a$ vaziyatdan $x=b$ vaziyatga ko'chirishda *F* kuchning bajargan ishi

$$A=F(b-a) \qquad (45.2)$$

formula yordamida topilishi ma'lum.

2. *F* kuch *M* nuqtaning vaziyatiga bog'liq ravishda o'zgarsin, ya'ni $[a,b]$ kesmada $F(x)$ uzluksiz funksiya bo'lsin. U holda *F* kuch bajargan *A* ishni quyidag'icha topiladi (199- chizma).

$[a,b]$ kesmani $a=x_0< x_1< x_2<... x_{i-1}< x_i<...< x_n=b$ nuqtalar yordamida *n* ta ixtiyoriy $[x_{i-1}; x_i]$ ($i=\overline{1,n}$) mayda qismlarga ajratib har bir $[x_{i-1}; x_i]$ bo'lakda bittadan ixtiyoriy z_i nuqta olamiz. Uzunligi $\Delta x_i = x_i - x_{i-1}$ bo'lgan $[x_{i-1}; x_i]$ mayda bo'lakda *F* kuch o'zgarmas va u $F(z_i)$ga teng deb faraz qilamiz. U holda (45.2) formulaga ko'ra *F* kuchning $[x_{i-1}; x_i]$ oraliqda bajargan ishi

$$A_i \approx F(z_i)\,\Delta x_i$$

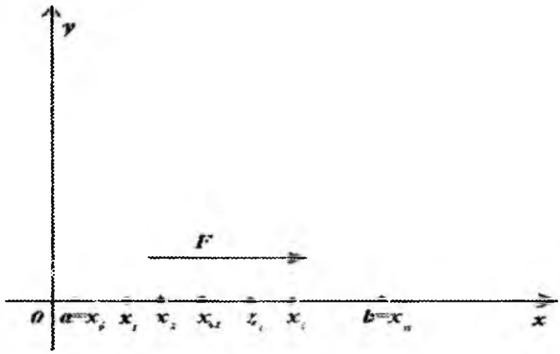
bo'ladi. Shunga o'xshash mulohazalarini har bir kesma uchun o'tkazib *F* kuchning $[a;b]$ kesmada bujargan ishi *A* ning taqribiy qiymati

$$A \approx \sum_{i=1}^{n} F(z_i)\Delta x_i$$

ni hosil qilamiz. Bu tenglikning o'ng tomonidagi yig'indi *[a;b]* kesmada uzluksiz *F(x)* funksiya uchun integral yig'indi bo'ladi. Shuning uchun u $\lambda = \max\Delta x_i \to 0$ da aniq limitga ega va *F(x)* funksiyadan *[a;b]* oraliq bo'yicha olingan aniq integralga teng, ya'ni

$$A = \int_a^b F(x)dx. \tag{45.3}$$

8-misol. Yer sathidan vertikal yo'nalishda *m* massali jismni *h* balandlikka chiqarish uchun zarur bo'lgan kuchning bajargan A ishi topilsin (200-chizma).



199-chizma.

Yechish. Yerning tortish kuchini *F,* massasini m_y, jismdan yerning markazigacha masofani *x* desak Nyuton qonuniga ko'ra

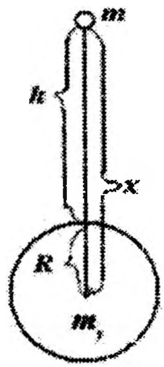
$$F = G\frac{m \cdot m_y}{x^2}$$

bo'ladi. Agar $Gm \cdot m_y = K$ belgilashni kiritsak $F(x) = \frac{K}{x^2}$ ga ega bo'lamiz, bunda $R \le x \le h + R$, *R-* yerning radiusi. $x = R$ da *F(R)*kuch jismning og'irlik kuchi $P=m\cdot g$ ga teng va $\frac{K}{x^2}=P$, $K=PR^2$, $F(x)=\frac{PR^2}{x^2}$.

Buni (45.3) formulaga qo'yib quyidagini hosil qilamiz.

$$A=\int_{R}^{R+h}F(x)dx=PR^2\int_{R}^{R+h}\frac{dx}{x^2}=-PR^2\frac{1}{x}\Big|_{R}^{R+h}=-\frac{PR^2}{R+h}+\frac{PR^2}{R}=\frac{PRh}{R+h}.$$

9-misol. Ikkinchi kosmik tezlik topilsin.



200-chizma.

Yechish. Jismning ikkinchi kosmik tezligini ya'ni (200-chizma) jism yerning tortish maydonidan planetalararo fazoga chiqishi uchun u qanday boshlang'ich tezlikka ega bo'lishi kerak degan savolga javob izlaymiz. 8-misolning natijasidan hamda undagi belgilashlardan foydalanamiz.

Jismning planetalararo fazoga chiqishi uni cheksiz balandlikka $(h\to\infty)$ chiqishni anglatadi. Shuning uchun

$$A=\frac{PRh}{R+h}$$

tenglikda $h\to\infty$ da limitga o'tib quyidagini hosil qilamiz.

$$\lim_{h\to\infty} A = \lim_{h\to\infty}\frac{PR}{\frac{R}{h}+1} = PR = mgR,$$

bunda g-jismning yer sathiga erkin tushish tezlanishi. Bu ish jismning kinetik energiyasining o‘zgarishi hisobiga amalga oshiriladi. Shuning uchun boshlang‘ich paytda jismning kinetik energiyasi shu ishdan kichik bo‘lmasligi, ya‘ni jismning boshlang‘ich tezligi v shunday bo‘lishi kerakki $\frac{mv^2}{2} \geq mgR$ bo‘lsin.

Bundan

$$v^2 \geq 2gR,$$

$$v \geq \sqrt{2gR} = \sqrt{2\cdot 10\cdot 6400000}\ m/s = 1{,}4\cdot 8000\ m/s = 11{,}2\ \ km/s.$$

Agar jismning boshlang‘ich tezligi *11,2 km/s* ga teng bo‘lsa jism parabola bo‘ylab harakatlanadi. Jismning boshlang‘ich tezligi *11,2 km/*s dan katta bo‘lganda jism giperbola bo‘ylab harakat qiladi. Jismning boshlang‘ich tezligi *11,2 km/s* dan kichik bo‘lganda u ellips bo‘ylab harakat qiladi. Shuning uchun bu holda jism yo yerga qulab tushadi yoki yerning sun‘iy yo‘ldoshiga aylanadi.

10-misol. Agar prujinani *1 sm* ga qisish uchun *10 N* kuch kerak bo‘lsa, uni *4 sm* ga qisish ga sarf bo‘ladigan F kuch bajaradigan ishni toping.

Yechish. Guk qonuniga muvofiq F kuch va x siljish o‘zaro $F=kx$ munosabat bilan bog‘langan (k-proporsionallik koeffitsienti). k ni masala shartidan topamiz: $x=1\ sm=0{,}01\ m$ da kuch $F=10\ N$, ya‘ni $10=\kappa\cdot 0{,}01$ bundan $\kappa=1000\ N/m$. Demak(45.3) ko‘ra

$$A = \int_0^{0,04} 1000x\,dx = 500x^2 \Big|_0^{0,04} = 500N/m\cdot 0{,}0016m^2 = 0{,}8(J).$$

11-misol. Uzunligi *1m*, kesimining radiusi *2mm* bo‘lgan mis simni *1mm* ga cho‘zishda bajarilgan ishni hisoblang.

Yechish. Uzunligi *lm* va kesimining yuzi Smm^2 bo'lgan simni *x m* ga cho'zish uchun zarur *F* kuch $F=E\frac{Sx}{l}$ formula orqali ifodalanadi, bu yerda *E*-elastiklik moduli. Mis uchun *E* ni *E=120000* N/mm^2 deb olamiz. U holda

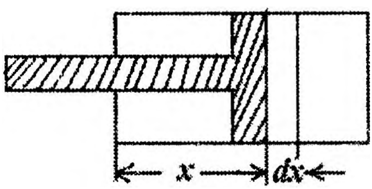
$$A=\int_0^{0,001}E\cdot\frac{Sx}{\ell}dx=\frac{ES}{\ell}\int_0^{0,001}xdx=\frac{E\cdot S}{\ell}\cdot\frac{x^2}{2}\Big|_0^{0,001}=\frac{ES}{2\ell}\cdot(0,001)^2.$$

Bunga $E=120000\ N/mm^2$, $S=2\pi R=2\pi\cdot 2=4\ \pi mm^2$, $\ell=1m$ qiymatlarni qo'yib bajarilgan ishni topamiz:

$$A=120000\cdot 4\pi\cdot\frac{0,000001}{2\cdot 1}=0,24\pi(J).$$

12-misol Ko'ndalang kesimining yuzi *S kv.* birlik bo'lgan harakatlanuvchi porshenga ega silindr gaz bilan to'ldirilgan. Gazning hajmi oshganda silindrda Boyl-Mariotta qonuni $pv=k=const$ saqlanadi deb hisoblab gazning bosim kuchi ta'sirida uning hajmi v_0 dan v_1 gacha o'zgarganda shu kuchning bajargan ishi *A* topilsin (gazning harorati o'zgarmaydi).

Yechish. *x(m)*-porshenning o'tgan masofasi bo'lsin (201-chizma).



201-chizma.

x juda kichik *dx* ga o'zgarganda gazning bosimi o'zgarmaydi hajmi v esa Δv ga o'zgaradi deb faraz qilamiz. U holda bosim kuchining *dx* kesmada bajargan ishi ΔA quyidagi taqribiy qiymat yordamida ifodalanadi, ya'ni

$\Delta A \approx P \cdot Sdx.$ $P = \frac{k}{v}$ va $Sdx = \Delta v$ (silindrning hajmi asosining yuzi S bilan balandligi dx ning ko'paytmasiga teng) ekanini hisobga olsak

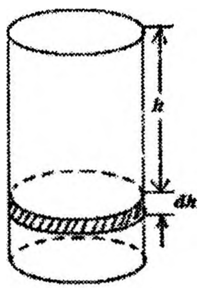
$$\Delta A \approx \frac{k}{v}\Delta v = k\frac{\Delta v}{v}$$

hosil bo'ladi. Bu yerdagi ΔA, Δv orttirmalarni dA, dv differensiallarga almashtirib

$$dA \approx k\frac{dv}{v}$$

tenglikka ega bo'lamiz. Buni v_0 dan v_1 gacha integrallab $A = k\ln\frac{v_1}{v_0}$ ni hosil qilamiz.

13-misol. Asosining radiusi $R=3m$, balandligi $P=5$ m bo'lgan silindrik idishdagi suvni tortib chiqarish uchun kerak bo'ladigan ishni hisoblang (202-chizma).



202-chizma.

Yechish. Birorta jismni ko'tarishga sarflanadigan kuchning bajargan A ishning kattaligi jismni ko'tarish balandligi h ga bog'liq bo'ladi, ya'ni

$$A=Ph$$

bu yerda P- jismning og'irligi.

Ma'lumki, idishdan h balandlikdagi suv qatlamini tortib chiqarish uchun sarf bo'ladigan kuchning bajargan ishi h ning funksiyasi, ya'ni $A(h)$ bo'ladi. h miqdor dh kattalikka ortganda suv hajmi $\Delta v=\pi R^2 dh$ ga (silindrning hajmi asosining yuzi bilan balandligining ko'paytmasiga teng), uning og'irligi P, $\Delta P=\pi \rho g R^2 dh$ $(\Delta P=\rho g \cdot \Delta v)$ kattalikka (bu yerda ρ-suvning zichligi, g erkin tushish tezlanishi), ish esa $\Delta A=\pi \rho g R^2 hdh$ kattalikka ortadi. ΔA orttirmani dA differensialga almashtirib

$$dA=\pi \rho g R^2 hdh$$

tenglikka ega bo'lamiz. Bu tenglikni $h=0$ dan $h=H$ gacha integrallab butun A ishni topamiz:

$$A=\int_0^H \pi\rho g R^2 hdh=\frac{\pi\rho g R^2}{2}\cdot h^2\Big|_0^H=\frac{\pi\rho g R^2 H^2}{2},$$

bunga $\rho g=1 tonna/m^3=10000\ N/m^3$ ekanini hisobga olib va $R=3m$, $H=5\ m$ qiymatlarni qo'ysak

$$A=\frac{1}{2}\pi\cdot 10000\cdot 9\cdot 25=1125000\ (J).$$

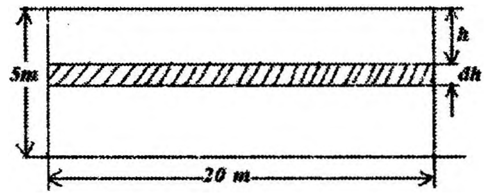
45.3. Suyuqlikning bosim kuchini hisoblash

Suyuqlikning bosim kuchini hisoblash uchun Paskal qonunidan foydalaniladi, unga ko'ra cho'kish (botish) chuqurligi h bo'lgan S yuzga suyuqlikning bosim kuchi

$$P=\rho g hS$$

ga teng, bu yerda ρ-suyuqlikning zichligi, $g=9{,}807$ erkin tushish tezlanishi.

14-misol. Vertikal to'g'on asosi $20m$ va balandligi $5m$ bo'lgan to'g'ri to'rtburchak shaklida (suvning sathi to'g'onning yuqori asisi bilan barobar), suvning butun to'g'onga bosim kuchini toping (203-chizma).



203-chizma.

Yechish. Paskal qonuniga muvofiq: $P=\rho g\, hS=9807\, hS(N)$ (suv uchun $\rho g=1000\cdot 9{,}807 N/m^3=9807 N/m^3$) ya'ni bosim kuchi h chuqurlikning birorta $P(h)$ funksiyasidan iborat. Eni juda kichik dh ga teng shtrixlanagan to'g'ri to'rtburchakni olib uni h chuqurlikda gorizontal joylashgan deb faraz qilamiz. U holda bu bo'lakchaga bo'lgan bosim kuchi

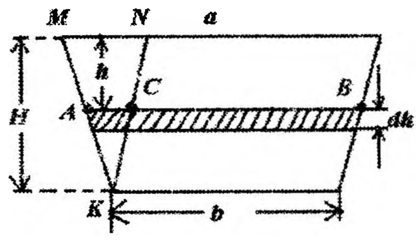
$$dh=9807h\cdot 20dh=9807\cdot 20hdh$$

bo'ladi. Buni 0 dan 5 gacha integrallab suvning butun to'g'onga bosim kuchini topamiz:

$$P=9807\cdot 20\int_0^5 hdh=9807\cdot 10\cdot h^2\Big|_0^5=$$

$$=9807\cdot 250(N)=2451750(N)=2{,}45(MN).$$

15-misol. Vertikal to'g'on teng yonli trapetsiya shaklida bo'lib, yuqori asosi $a=6{,}4$ m, pastki asosi $b=4{,}8$ m, balanligi esa $H=3$ m va suvning sathi to'g'onning yuqori asosi bilan barobar. Suvning butun to'g'onga bosim kuchini toping (204-chizma).



204-chizma.

Yechish. Trapetsiyaning shtrixlangan bo'lakchasi h chuqurlikda gorizontal joylashgan va u tomonlari AB va dh bo'lgan to'g'ri to'rtburchakdan iborat deb faraz qilamiz. U holda bu bo'lakka bo'lgan suvning bosimi.

$$dP=9807hABdh=9807h(AC+CB)dh=9807h(AC+b)dh \quad (N)$$

bo'ladi. AC ni KAC va KMN uchburchaklarning o'xshashligidan topamiz:

$$\frac{AC}{MN}=\frac{H-h}{H}, \quad \frac{AC}{a-b}=\frac{H-h}{H}, \qquad AC=\frac{a-b}{H}(H-h).$$

Bu ifodani h bo'yicha 0 dan H gacha integrallab, butun to'g'onga ta'sir etayotgan bosim kuchini topamiz:

$$P=9807\cdot\frac{1}{H}\int_0^H h(aH-h(a-b)dh)=9807\cdot\frac{1}{H}\int_0^H\left[aHh-(a-b)h^2\right]dh=$$

$$=9807\cdot\frac{1}{H}\left(\frac{aHh^2}{2}-\frac{(a-b)h^3}{3}\right)\Bigg|_0^H=9807\cdot\frac{1}{H}\left(\frac{aH^3}{2}-\frac{(a-b)H^3}{3}\right)=$$

$$=9807\cdot\frac{H^3}{H}\left(\frac{3a-2a+2b}{6}\right)=9807\cdot\frac{H^2(a+2b)}{6}.$$

Bunga $H=3$ m, $a=6,4$ m, $b=4,8$ m qiymatlarni qo'yib, topamiz:

$$P=9807\cdot\frac{9(6{,}4+4{,}8\cdot2)}{6}=9807\cdot24=235368 \quad (N).$$

45.4. Kinetik energiya

Massasi m ga, tezligi v ga teng bo'lgan moddiy nuqtaning kinetik energiyasi deb

$$k=\frac{mv^2}{2}$$

kattalikka aytiladi.

Massalari $m_1, m_2, ..., m_n$, tezliklari mos ravishda $v_1, v_2, ..., v_n$, larga teng bo'lgan n ta moddiy nuqtalar sistemasining kinetik energiyasi

$$K = \sum_{i=1}^{n} \frac{m_i v_i^2}{2}$$

ga tengdir.

Moddiy jism (figura)ning kinetik energiysini ham yuqorida qaralgan masalalarni yechishda foydalanilgan usuldan foydalanib topamiz, ya'ni berilgan jismni n ta kichik (elementar) qismlarga ajratib ularni moddiy nuqtalar sistemasi deb qaraymiz va ularni kinetik energiyalarini jamlab qandaydir funksiyaning integral yig'indisiga ega bo'lamiz. Unda limitga o'tib, qiymati jismning izlanayotgan kinetik energiyasiga teng bo'lgan aniq integralni hosil qilamiz.

16-misol. Massasi M va radiusi R bo'lgan disk uning markazidan disk tekisligiga perpendikulyar bo'lib o'tgan o'q atrofida ω burchak tezlik bilan aylanyapti. Uning kinetik energiyasini hisoblang. (205-chima)

Yechish. Diskni radiuslari $0<r_1<r_2<r_3<...<r_{i-1}<r_i<...<r_n=R$ bo'lgan aylanalar yordamida n ta ixtiyoriy halqalarga ajratamiz. Qalinligi (kengligi) $\Delta r_i = r_i - r_{i-1}$ $(i = \overline{1,n})$ bo'lgan halqani qaraymiz. Bu halqaning massasi

$$\Delta m_i = \rho \Delta s_i = \rho\pi(r_i^2 - r_{i-1}^2) = \rho\pi(r_i + r_{i-1})(r_i - r_{i-1}) =$$

$$= 2\pi\rho \frac{r_i + r_{i-1}}{2} \Delta r_i = 2\pi\rho \overline{r}_i \Delta r_i$$

bunda $\rho = \dfrac{M}{\pi R^2}$ -diskning zichligi, $\overline{r}$ $[r_i, r_{i-1}]$ kesmaning o'rtasi. U holda

$$\Delta m_i = 2\pi \overline{r}_i \cdot \frac{M}{\pi R^2} \Delta r_i = \frac{2\overline{r}_i M}{R^2} \Delta r_i.$$

Δm_i massaning chiziqli tezligi $v_i = \overline{r_i}\omega$ ga teng. Demak elementar kinetik energiya quyidagiga teng bo'ladi:

$$\Delta K_i = \frac{v_i^2 \Delta m_i}{2} = \frac{(\overline{r_i}\omega)^2}{2} \cdot \frac{2\overline{r_i}M}{R^2}\Delta r_i = \frac{\omega^2 M}{R^2}\overline{r_i}^3 \Delta r_i .$$

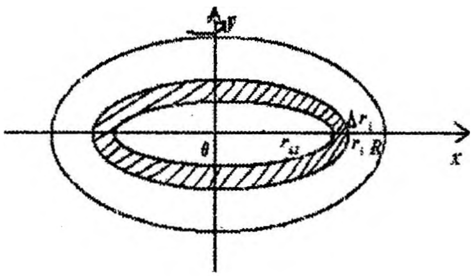
Barcha elementar kinetik energiyalarni jamlab

$$K \approx \sum_{i=1}^{n} \frac{\omega^2 M}{R^2}\overline{r_i}^3 \Delta r = \frac{\omega^2 M}{R^2}\sum_{i=1}^{n}\overline{r_i}^3 \Delta r$$

ga ega bo'lamiz. Bunda $\max \Delta r_i \to 0$ da limitga o'tsak

$$K = \frac{\omega^2 M}{R^2}\int_0^R r^3 dr = \frac{\omega^2 M}{R^2} \cdot \frac{r^4}{4}\Big|_0^R = \frac{\omega^2 MR^2}{4}$$

hosil bo'ladi.



205-chizma.

O'z-o'zini tekshirish uchun savollar

1.O'zgarmas tezlikda yo'lni topish formulasini yozing.
2. Aniq integral yordamida yo'lni topish formulasini yozing.
3. Tekis sekinlanuvchan harakatning tezligi qanday topiladi?
4. Tekis tezlanuvchan harakatning tezligi qanday topiladi?
5. Nyutonning butun olam tortishish qonuni ayting?
6. O'zgaruvchan kuchning bajargan ishini topish formulasini yozing.
7. Guk qonunini ayting.

8. Paskal qonuni nimadan iborat?
9. Moddiy nuqtaning kinetik energiyasi nima va u qanday topiladi?
10. Moddiy jismning kinetik energiyasi qanday topiladi?

Mustaqil yechish uchun mashqlar

1. Nuqtaning tezligi $v=(100+8t)$ *m/sek.* Bu nuqta $[0;10]$ vaqt oralig'ida qanday masofani bosadi. Javob: *1400 m.*

2. Nuqtaning harakat tezligi $v=t\cdot e^{-0,01t}$ *m/sek* ga teng. Harakat boshlangandan to'la to'xtagunga qadar nuqta bosib o'tgan yo'lni toping. Javob: *10 km.*

3. Agar prujinani *1 sm* ga qisish uchun *10 N.* kuch kerak bo'lsa, prujinani *8 sm* ga qisish uchun sarf bo'ladigan F kuchning bajaradigan ishini toping. Javob: *3,2 (J).*

4. Prujinani *0,05 m* ga qisish uchun *25 J* ish sarflanadi. Prujinani *0,1 m* ga qisish uchun qancha sarflanadi? Javob: *100 J.*

5. Asosi *0,2 m* va balandligi *0,4 m* bo'lgan uchburchakli plastinka suvga shunday tik botirilganki uning uchi suvning sathida yotib asosi unga parallel. Suvning plastinkaga bosim kuchini toping. Javob: *104,6 (N).*

6. Markazi suyuqlikka (suyuqlikning zichligi δ ga teng) h chuqurlikda botirilgan $2a$ va $2b$ o'qli vertikal ellipsga (ellipsning katta o'qi $2a$ suyuqlik sathiga parallel, $h\geq b$) suyuqlikning bosim kuchini toping. Javob: $P=9,81\delta ab\pi h$.

7. Diametr atrofida minutiga n marta aylanayotgan M massali va R radiusli yarim aylananing kinetik energiyasini hisoblang.

Javob: $\dfrac{\pi^2 n^2 R^2 M}{3600}$.

46. BIR NECHA O‘ZGARUVCHINING FUNKSIYASI

46.1. Bir necha o‘zgaruvchining funksiyasi tushunchasi

Radiusi R ga teng doiraning S yuzi $S=\pi R^2$ formula yordamida topilishni bilamiz. Bunda doira radiusining o‘zgarishi uning yuzini o‘zgarishga majbur etadi. Boshqacha aytganda doiraning yuzi S faqat bitta o‘zgaruvchi R ning funksiyasidir.

Tomonlarining uzunligi x va y ga teng bo‘lgan to‘g‘ri to‘rtburchakning yuzi S ushbu formula bilan ifodalanadi:

$$S=xy.$$

To‘g‘ri to‘rtburchakning tomonlari x va y ning o‘zgarishi uning yuzi S ni o‘zgarishiga olib keladi. x va y ning har bir juft (x,y) qiymatiga S yuzning aniq qiymati mos keladi. S ikki o‘zgaruvchining funksiyasidir.

Qirralarining uzunliklari x,y va z ga teng bo‘lgan to‘g‘ri burchakli parallelepipedning v hajmi

$$v=xyz$$

formula yordamida topilishi ayon. Qirralarning uzunligini o‘zgarishi hajmni o‘zgarishga majbur etadi, ya‘ni x,y va z ning har bir (x,y,z) uchlik qiymatiga v hajmning aniq qiymati mos keladi. v hajm uchta o‘zgaruvchi x,y,z ning funksiyasidir.

Jismning harorati qaralsa u jismning har xil nuqtalarida har xil, ya‘ni shu nuqtaning koordinatalari x,y,z larga bog‘liq bo‘ladi. Harorat t vaqtga ham bog‘liq ekanligini hisobga olsak harorat to‘rtta o‘zgaruvchi x,y,z va t ga bog‘liq bo‘ladi. Boshqacha aytganda harorat to‘rt o‘zgaruvchining funksiyasidir.

Keltirilgan misollardan ko‘rinib turibdiki ko‘pgina hodisalarni o‘rganishda ikki, uch va undan ortiq o‘zgaruvchilarning funksiyasi bilan ish ko‘rishga to‘g‘ri keladi.

Shuning uchun bir necha o‘zgaruvchining funksiyasi tushunchasini kiritamiz va uni o‘rganishga kirishamiz.

1-ta'rif. Agar bir-biriga bog'liq bo'lmagan ikki o'zgaruvchi x va y ning biror D o'zgarish sohasidagi har bir juft (x,y) qiymatiga biror qoida bilan E to'plamdagi z o'zgaruvchining aniq bir qiymati mos kelsa, u holda D sohada **ikki o'zgaruvchining funksiyasi** z aniqlangan deyiladi.

x va y **erkli o'zgaruvchilar** yoki **argumentlar**, z esa **erksiz o'zgaruvchi** yoki **funksiya** deb ataladi.

Ikki o'zgaruvchining funksiyasi

$z=z(x,y)$, $z=f(x,y)$, $z=\varphi(x,y)$, $z=F(x,y)$ va hokazo ko'rinishda belgilanadi.

D to'plam funksiyaning **aniqlanish sohasi** deyiladi. z o'zgaruvchini qiymatlari to'plami E funksiyaning **o'zgarish sohasi** (qiymatlar to'plami) deyiladi. $z=f(x,y)$ fnuksiyaning argumentlarning $x=x_0$, $y=y_0$ tayin (aniq) qiymatlariga mos z_0 xususiy qiymati $z_0 = z\Big|_{\substack{x=x_0\\ y=y_0}}$ yoki $z_0=f(x_0, y_0)$ kabi yoziladi. Masalan, $x=-1$, $y=2$ da

$z=x^3+y^2$ funksiyaning qiymati $z\Big|_{\substack{x=-1\\ y=2}} = (-1)^3 + 2^2 = 3$ bo'ladi.

Bir o'zgaruvchining funksiyasi kabi ikki o'zgaruvchining funksiyasi ham, umuman aytganda argumentlar x va y ning barcha qiymatlarida mavjud bo'lavermaydi.

Masalan, $z=\sqrt{1-x^2-y^2}$ funksiya faqatgina x va y ning $1-x^2-y^2 \geq 0$ tengsizlikni qanoatlantiradigan qiymatlaridagina mavjud.

2-ta'rif. $z=f(x,y)$ funksiya aniqlangan x va y ning (x,y) juft qiymatlarining to'plami funksiyaning **aniqlanish sohasi** yoki **mavjudlik sohasi** deb ataladi. Ikki o'zgaruvchi funksiyasining aniqlanish sohasi geometrik tarzda ko'rgazmali tasvirlanadi.

Geometrik nuqtai nazardan to'g'ri burchakli koordinatalar sistemasida haqiqiy sonlarning har bir tartiblangan (x,y) juftiga $0xy$ tekislikning x va y koordinatali yagona P nuqtasi mos keladi; aksincha 0xy tekislikning har bir $P(x,y)$ nuqtasiga haqiqiy sonlarning yagona (x,y) jufti mos keladi. Bu munosabat bilan ikki o'zgaruvchining funksiyasini $P(x,y)$ nuqtaning funksiyasi sifatida qarash mumkin.

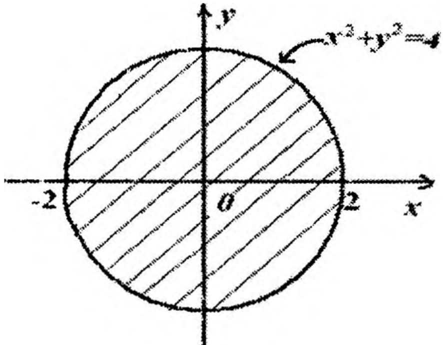
Shunday qilib $z=f(x,y)$ o'rniga $z=f(P)$ yozish mumkin. U holda ikki o'zgaruvchi funksiyasining aniqlanish sohasi D tekislikning biror nuqtalari to'plami yoki butun Oxy tekislik bo'ladi.

1-misol. $z=x^2+y^2$ funksiyaning aniqlanish va o'zgarish sohasi topilsin.

Yechish. Funksiya x va y ning barcha qiymatlarida aniqlangan, ya'ni butun Oxy tekislik funksiyaning aniqlanish sohasi. Funksiyaning qiymatlar sohasi $E=[0;\infty)$.

2-misol. $z=\sqrt{4-x^2-y^2}$ funksiyaning aniqlanish va o'zgarish sohalari topilsin.

Yechish. Bu funksiyaning aniqlanish sohasi $\sqrt{4-x^2-y^2}$ ifoda aniqlangan (ma'noga ega) nuqtalar to'plamidan ya'ni x va y ning $4-x^2-y^2\ge 0$ yoki $4\ge x^2+y^2$ tengsizlikni qanoatlantiruvchi qiymatlaridan iborat. $x^2+y^2=4$ markazi koordinatalar boshida bo'lgan radiusi 2 ga teng aylana tenglamasi ekaligini hisobga olsak $x^2+y^2\le 4$ shu aylana bilan chegaralangan sohani, ya'ni doirani ifodalaydi. Aylananing nuqtalari ham funksiyaning aniqlanish sohasiga tegishli. Bu yerda kvadrat ildiz ildiz ostidagi ifoda nomanfiy bo'lganda ma'noga ega ekanligi hisobga olindi. Funksiyaning qiymatlar sohasi $E=[0,2]$ kesmadan iborat (206-chizma).



206-chizma.

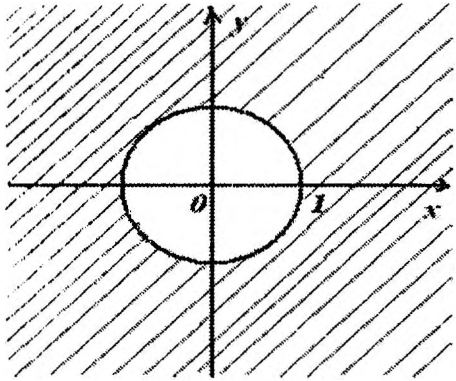
3-misol. $z=\dfrac{1}{\sqrt{x^2+y^2-1}}$ funksiyaning aniqlanish va o'zgarish sohalari topilsin.

Yechish. Funksiyaning aniqlanish sohasi $\dfrac{1}{\sqrt{x^2+y^2-1}}$ ifoda aniqlangan (ma'noga ega) nuqtalar to'plami, ya'ni $x^2+y^2-1>0$ yoki $x^2+y^2>1$ bajariladigan nuqtalar to'plami bo'ladi. Bu to'plamga $0xy$ tekislikning markazi koordinatalar boshida bo'lib radiusi 1 ga teng aylanadan tashqarida yotgan barcha nuqtalari tegishli (207-chizma). Bu yerda kasrning maxraji noldan farqli bo'lganda u ma'noga ega ekanligi hisobga olindi. Funksiyaning qiymatlar sohasi $E=(0,+\infty)$ intervaldan iborat.

4-misol. $z=\lg\sqrt{x^2+y^2-1}$ funksiyaning aniqlanish va o'zgarish sohalari topilsin.

Yechish. Logarifmik funksiya faqatgina argumentning musbat qiymatlarida ma'noga ega. Shunga ko'ra berilgan funksiya ham x va y ning $x^2+y^2-1>0$ yoki $x^2+y^2>1$ tengsizlikni qanoatlantiruvchi qiymatlarida aniqlangan. $x^2+y^2=1$ tenglama markazi (0;0) nuqtada bo'lib radiusi 1 ga teng aylanani ifodalaydi. $x^2+y^2>1$ tengsizlik $0xy$ tekislikning shu aylanadan tashqarida yotgan nuqtalari to'plamini ifodalaydi. $x^2+y^2=1$ bo'lganda $lg0$ bo'lib u ma'noga ega emas. Demak aylananing nuqalari funksiyaning aniqlanish sohasiga tegishli emas. $x^2+y^2<1$ bo'lganda kvadrat ildiz ostidagi ifoda manfiy bo'lib kvadrat ildiz va u bilan birga logarifmik funksiya ham ma'noni yo'qotadi.

Shunday qilib, berilgan funksiyaning aniqlanish sohasi $0xy$ tekislikning markazi koordinatalar boshida bo'lib radiusi 1 ga teng doirasidan tashqarida yotgan nuqtalari to'plamidan iborat ekan (207-chizma). Funksiyaning qiymatlar sohasi $E=(-\infty;+\infty)$ intervaldan iborat bo'lishi ko'rinib turibdi.



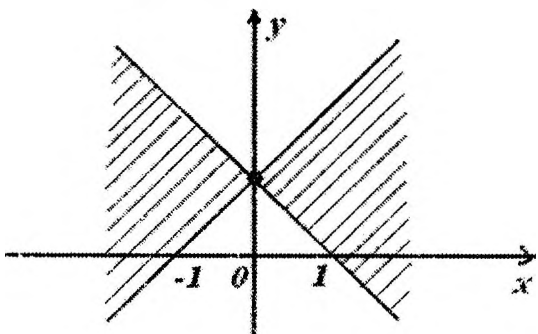
207-chizma.

5-misol. $z=\arcsin\frac{y-1}{x}$ funksiyaning aniqlanish sohasi topilsin.

Yechish. $y=\arcsin x$ funksiya [-*1*; *1*] kesmada aniqlangan. Shunga ko'ra

$$-1\le\frac{y-1}{x}\le 1$$

va bundan $x<0$ bo'lganda $-x\ge y-1\ge x$; $x+1\le y\le 1-x$ ga $x>0$ bo'lganda $-x\le y-1\le x$; $1-x\le y\le x+1$ tengsizliklarga ega bo'lamiz. $x=0$ da funksiya aniqlanmagan.



208-chizma.

Shunday qilib, $y=x+1$ va $y=1-x$ to‘g‘ri chiziqlar hosil qilgan o‘ng va chap vertikal burchaklar berilgan funksiyaning aniqlanish sohasi bo‘lar ekan, bunda to‘g‘ri chiziqlarning kesishishi nuqtasi (0;1)dan tashqari barcha nutalari ham funksiyaning aniqlanish sohasiga tegishli (208-chizma).

Bir o‘zgaruvchining funksiyasi kabi ikki o‘zgaruvchining funksiyasi ham bir necha usullar bilan berilishi mumkin. Biz asosan analitik usulda berilgan funksiyani qaraymiz.

Dekartning fazodagi koordinatalar sistemasi $0xyz$ ni qaraganda fazoning ixtiyoriy P nuqtasiga shu nuqtaning koordinatalari- haqiqiy sonlarning tartiblangan uchligi (x,y,z) mos kelishini va istalgan tartiblangan haqiqiy sonlarning uchligi (x,y,z) ga $0xyz$ fazoning $P(x,y,z)$ nuqtasi mos kelishini ko‘rgan edik. Shuning uchun ikki o‘zgaruvchining funksiyasi $z=f(P)$ da $0xy$ tekisligining $P(x,y)$ nuqtalari to‘plami o‘rnida $0xyz$ fazoning biror $P(x,y,z)$ nuqtalari to‘plami qaralsa uch o‘zgaruvchining funksiyasi $u=f(P)$ yoki $u=f(x,y,z)$ hosil bo‘ladi. Uch o‘zgaruvchi funksiyasining aniqlanish sohasi butun uch o‘lchovli $0xyz$ fazodan yoki uning biror qismidan iborat bo‘ladi.

Masalan, $u=x^2+y^2+z^2$ funksiya butun $0xyz$ fazoda aniqlangan. $u=\ln(4-x^2-y^2-z^2)$ funksiya esa fazoning markazi koordinatalar boshida bo‘lib radiusi 2 ga teng sfera bilan chegaralangan (shar) qismida aniqlangan. Sferaning nuqtalari funksiyaning aniqlanish sohasiga tegishli emas.

Yuqoridagi kabi mulohaza yuritib to‘rt o‘zgaruvchining funksiyasi $u=f(x,y,z,t)$ ga kelamiz. Bu holda haqiqiy sonlarning tartiblangan to‘rtligi (x,y,z,t) to‘rt o‘lchovli fazo deb ataluvchi fazo nuqtasining koordintalarini tashkil etadi. To‘rt o‘zgaruvchi funksiyasining aniqlanish sohasi butun to‘rt o‘lchovli fazodan yoki uning biror qismidan iborat bo‘ladi.

n ta haqiqiy sonlarning tartiblangan $(x_1,x_2,...,x_n)$ sistemalari to‘plami D ni qaraymiz, bunda n biror natural son.

3-ta’rif. Agar D to‘plamning har bir tartiblangan $(x_1,x_2,...,x_n)$ sistemasiga biror qoida yordamida aniq z son mos qo‘yilsa, u holda z

n ta $x_1, x_2, ..., x_n$ o'zgaruvchilarning D sohasida aniqlangan funksiyasi deb ataladi va $z = f(x_1, x_2, ..., x_n)$ kabi yoziladi.

Agar n o'zgaruvchining f va φ funksiyalari bitta $(x_1, x_2, ..., x_n)$ nuqtalar to'plami D da aniqlangan bo'lsa, u holda shu to'plamda aniqlangan ularning yig'indisi, ayirmasi, ko'paytmasi va maxraj noldan farqli bo'lganda bo'linmasi haqida gapirish mumkin.

Ma'lumki har bir haqiqiy son sonlar o'qining nuqtasi sifatida tasvirlanadi, ya'ni haqiqiy sonlar to'plami bir o'lchovli real (geometrik tasvirlanadigan) fazo-sonlar o'qidir.

Haqiqiy sonlarning tartiblangan har bir (x,y) juftligi tekislikning aniq nuqtasi sifatida tasvirlanadi, ya'ni haqiqiy sonlarning tartiblangan juftligi to'plami ikki o'lchovli real fazo-tekislkdir.

Haqiqiy solarning tartiblangan uchligi (x,y,z) fazoning aniq nuqtasi sifatida geometrik tasvirlanadi, ya'ni haqiqiy sonlarning tartiblagan barcha uchliklari to'plami uch o'lchovli real (geometrik tasvirlanadigan) fazodir.

Shuningdek, haqiqiy sonlarning tartiblangan $(x_1, x_2, ..., x_n)$ sistemalar to'plami n o'lchovli fazo deb ataladi va R^n kabi belgilanadi.

$n > 3$ bo'lganda $(x_1, x_2, ..., x_n)$ tartiblangan sonlar sistemasi n o'lchovli fazoning nuqtasi sifatida geometrik tasvirlanadigan real n o'lchovli fazo mavjud emas.

Noreal (geometrik tasvirlanmaydigan) bo'lsada n $(n>3)$ o'lchovli fazo mavjud va sonlarning $(x_1, x_2, ..., x_n)$ sistemasi uning nuqtasini ifodalaydi deb faraz qilamiz.

Matematiklar tomonidan o'ylab topilgan noreal n o'lchovli fazo ham ularga uch o'lchovli real fazodan kam xizmat qilmayapti.

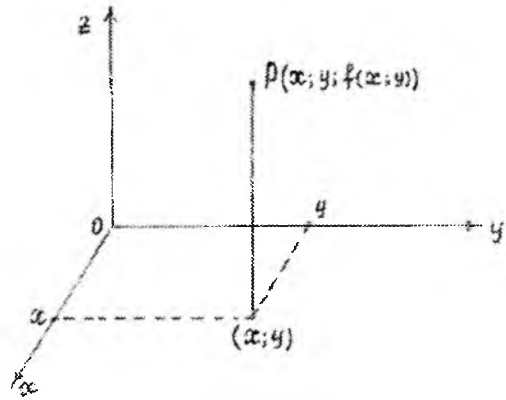
n o'zgaruchining funksiyasining aniqlanish sohasi n o'lchovli fazodan yoki uning biror qismidan iborat bo'lganligi sababli to'rt va undan ortiq o'zgaruvchili funksiyalarning aniqlanish sohalarini geometrik tasvirlab bo'lmaydi.

46.2. Ikki o‘zgaruvchi funksiyasining geometrik tasviri

Oxy tekislikdagi *D* sohada aniqlangan

$$z=f(x,y) \qquad (46.1)$$

funksiyani va *Oxyz* to‘g‘ri burchakli Dekart koordinatalari sistemasini qaraymiz (209-chizma),



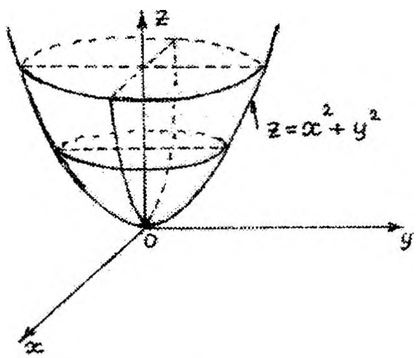
209-chizma.

bunda *D* soha butun *Oxy* tekislikdan iborat bo‘lishi ham mumkin. *D* sohaning har bir *(x,y)* nuqtasidan *Oxy* tekislikka perpendikulyar to‘g‘ri chiziq o‘tkazamiz va unda *f(x,y)* ga teng kesma ajratamiz. U holda fazoda koordinatalari *x,y*, *z=f(x,y)* bo‘lgan *P* nuqtani hosil qilamiz.

Koordinatalari *z=f(x,y)* tenglamani qanoatlantiradigan *P* nuqtalarning geometrik o‘rni ikki o‘zgaruvchi funksiyasi *z=f(x,y)* ning grafigi deb ataladi. (46.1) tenglama fazoda biron sirtni aniqlashi mumkin.

Demak ikki o‘zgaruvchi funksiyasining grafigi, *Oxy* tekislikdagi proeksiyasi funksiyaning aniqlanish sohasi *D* dan iborat bo‘lgan sirt bo‘ladi. *D* sohaning ixtiyoriy nuqtasidan *Oxy* tekislikka perpendikulyar o‘tkazilgan to‘g‘ri chiziq *z=f(x,y)* sirt bilan bir martadan ortiq kesishmaydi. Masalan, $z-3x+6y-7=0$ tenglama tekislik tenglamasi ekanini bilamiz. Demak bu tekislik $z=3x-6y+7$ ikki o'zgaruvchi funksiyasining grafigi.

Shuningdek $x^2+y^2+z^2=R^2$ tenglama markazi koordinatalar boshida bo'lgan radiusi R ga teng sferani tenglamasi edi. Demak sfera $z=\sqrt{R^2-x^2-y^2}$ va $z=-\sqrt{R^2-x^2-y^2}$ funksiyalarni grafiklarini birlashmasidan iborat, ya'ni sferaning Oxy tekislikdan pastda yotgan yarmi $z=-\sqrt{R^2-x^2-y^2}$ funksiyaning grafigini, sferaning Oxy tekislikdan yuqorida yotgan yarmi $z=\sqrt{R^2-x^2-y^2}$ funksiyaning grafigini ifodalaydi.



210-chizma.

Umuman olganda ikki o'zgaruvchi funksiyasining grafigini chizish unchalik oson ish emas. Uni ikkinchi tartibli sirtlarni chizishda foydalanilgan parallel kesimlar usulidan foydalanib, ya'ni sirtni koordinatalar tekisligiga parallel tekisliklar bilan kesilganda kesimda hosil bo'lgan chiziqqa qarab sirt haqida biror to'xtamga kelish mumkin.

Masalan, $z=x^2+y^2$ funksiyaning grafigini chizish uchun sirtni Oxy tekislikka parallel $z=h$ $(h>0)$ tekislik bilan kesganda kesimda aylana va Oyz, Oxz tekisliklariga parallel tekisliklar bilan kesganda kesimda parabola hosil bo'lishiga qarab $z=x^2+y^2$ funksiyaning grafigi 210–

chizmada tasvirlangan aylanish paraboloididan iborat bo'lishiga iqror bo'lamiz.

Agar A,B,C berilgan o'zgarmas sonlar bo'lganda $z = Ax + By + C$ funksiyani qarasak bu funksiya ikkita o'zgaruvchi x,y ning chiziqli funksiyasi bo'lib y butun Oxy tekislikda aniqlangan, grafigi tekislikdan iborat.

Shunga o'xshash A,B,C,D berilgan o'zgarmas sonlar bo'lganda $U = Ax + By + Cz + D$ funksiya uchta o'zgaruvchi x,y,z larning chiziqli funksiyasi bo'lib u uch o'lchovli $Oxyz$ fazoda aniqlangan, grafigini geometrik tasvirlab bo'lmaydi.

Chiziqli funksiya tushunchasini n ta o'zgaruvchilar uchun ham umumlashtirish mumkin. a_i $(i = \overline{1,n})$, b berilgan o'zgarmas sonlar bo'lganda

$$U = a_1x_1 + a_2x_2 + \ldots + a_nx_n + b$$

funksiya n ta o'zgaruvchi $x_1, x_2, \ldots, x_n$ larning **chiziqli funksiyasi** deb ataladi. Bu funksiya n o'lchovli fazoning barcha nuqtalarida aniqlangan. $n \geq 4$ bo'lganda bu funksiyaning aniqlanish sohasini, $n \geq 3$ bo'lganda uning grafigini geometrik tasvirlab bo'lmaydi.

Agar

$$z = f(x,y) = \begin{cases} 0 & y \geq 0 \text{ bo'lganda}, \\ 1 & y < 0 \text{ bo'lganda} \end{cases}$$

funksiyani qarasak bu funksiya Oxy tekislikda aniqlangan, grafigi $Oxyz$ fazoda joylashgan o'zaro parallel ikkita yarim tekisliklardan iborat.

Shuni aytish joizki bir o'zgaruvchining funksiyasi kabi har qanday ikki o'zgaruvchining funksiyasi ham grafikka ega bo'lavermaydi. Masalan

$$f(x,y) = \begin{cases} 1, \text{ agar } x,y \text{ ratsional son bo'lsa}, \\ 0, \text{ agar } x,y \text{ larning kamida bittasi irratsional son bo'lsa} \end{cases}$$

funksiya butun Oxy tekislikda aniqlangan, qiymatlari to'plami $E=\{0,1\}$. Bu funksiya grafikka ega emas.

Izoh. Uch va undan ortiq o'zgaruvchining funksiyasini fazoda grafik yordamida tasvirlash mumkin emas.

O'z-o'zini tekshirish uchun savollar

1. Bir o'zgaruvchi funksiyasini ta'riflang?
2. Bir o'zgaruvchi funksiyasining aniqlanish va o'zgarish sohalari nima?
3. Asosiy elementar funksiyalarning aniqlanish va o'zgarish sohalarini ayting.
4. Funksiyaning berilish usullarini ayting.
5. Grafigi mavjud bo'lmagan funksiyaga misol keltiring.
6. Bir necha o'zgaruvchi funksiyasini ta'riflang?
7. Bir necha o'zgaruvchi funksiyasining aniqlash va o'zgarish sohalari nima?
8. Ikki o'zgaruvchi funksiyasining aniqlanish sohasi nimadan iborat bo'ladi?
9. Uch o'zgaruvchining funksiyasini aniqlanish sohasi nimadan iborat?
10. Ikki o'zgaruvchi funksiyasining geometrik tasviri nimadan iborat?

Mustaqil yechish uchun mashqlar

Quyidagi funksiyalarning aniqlanish sohasi geometrik tasvirlansin.

1. $z=\ln(y-x^2)$.
2. $z=\arcsin(x^2+y^2-5)$.
3. $z=\sqrt{x^2+y^2-25}$.
4. $z=\sqrt{x^2+y^2-4}+\sqrt{16-x^2-y^2}$.
5. $u=\ln(z-x^2-y^2)$.
6. $z=\sqrt{x^2+y^2+z^2-9}$.
7. $u=\dfrac{x^2+y^2+z^2}{\sqrt{16-x^2-y^2-z^2}}$.

47. BIR NECHA O'ZGARUVCHI FUNKSIYANING LIMITI VA UZLUKSIZLIGI

47.1. Bir necha o'zgaruvchi funksiyaning limiti

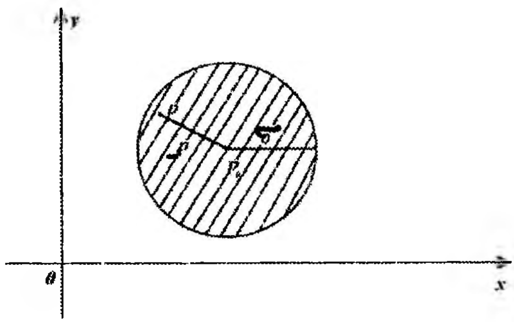
Funksiyaning limiti tushunchasini qarashdan oldin, berilgan nuqtaning **δ-atrofi** tushunchasini kiritamiz. Bir o'zgaruvchining funksiyasi qaralganda $x=x_0$ nuqtaning ∂-atrofi deganda $(x_o-\delta, x_o+\delta)$ interval tushunilar edi.

1-ta'rif. $0xy$ tekislikning koordinatalari

$$\rho(P;P_o)=\sqrt{(x-x_o)^2+(y-y_o)^2}<\delta$$

tengsizlikni qanoatlantiruvchi $P(x,y)$ nuqtalari to'plami $P_o(x_o,y_o)$ nuqtaning **δ-atrofi** deyiladi.

Boshqacha aytganda P_o nuqtaning δ-atrofi bu markazi P_o nuqtada bo'lgan ∂ radiusli doiraning ichki nuqtalaridir (aylananing nuqtalari atrofga tegishli emas) (211-chizma).



211-chizma.

Fazodagi $P_o(x_o,y_o,z_o)$ nuqtaning **δ-atrofi** markazi P_o nuqtada bo'lib radiusi δ ga teng

$$\rho(P;P_o)=\sqrt{(x-x_o)^2+(y-y_o)^2+(z-z_o)^2}<\delta$$

sharning $P(x,y,z)$ nuqtalaridan iborat bo'ladi. Sharning sirti-sferaning nuqtalari nuqtaning atrofiga tegishli bo'lmaydi. Bunda $\rho(P,P_o)$ orqali P_o va P nuqtalar orasidagi masofa belgilangan.

n o'lchovli ($n>3$ da) fazoda $P_o(x_{10}, x_{20}, \dots x_{no})$ nuqtaning δ-atrofi markazi P_0 nuqtada bo'lib radiusi δ ga teng $\rho(P, P_0) = \sqrt{\sum_{k=1}^{n}(x_k - x_{k0})^2} < \delta$ n o'lchovli sharning sirtida yotmagan $P(x_1, x_2, \dots, x_n)$ nuqtalaridan iborat bo'ladi.

Faraz qilaylik $z=f(x,y)$ funksiya $P_o(x_0, y_0)$ nuqtaning biror atrofida aniqlangan bo'lsin (P_o nuqtaning o'zida aniqlanmagan bo'lishi ham mumkin).

2-ta'rif. Agar ixtiyoriy $r>0$ son uchun shunday $\delta>0$ son topilsaki,

$$\rho\ (P, P_o) < \delta$$

tengsizlikni qanoatlantiruvchi P_o dan farqli barcha $P(x,y)$ nuqtalar uchun

$$|f(x,y)-A| < \varepsilon \quad (\text{yoki } |f(P)-A| < \varepsilon)$$

tengsizlik bajarilsa, u holda A o'zgarmas son $z=f(x,y)$ funksiyaning $P_o(x_0, y_0)$ nuqtadagi yoki $P(x,y) \to P_o(x_0, y_0)$ dagi limiti deyiladi va

$$\lim_{p \to p_o} f(P) = A \quad \text{yoki} \quad \lim_{\substack{x \to x_0 \\ y \to y_o}} f(x,y) = A$$

ko'rinishda yoziladi.

Uch va undan ortiq o'zgaruvchi funksiyasining limiti ham shunga o'xshash ta'riflanadi.

1-misol. $f(x,y) = \dfrac{x^3+y^3}{x^2+y^2} (x^2+y^2 \neq 0)$ funksiyaning $0(o,o)$ nuqtadagi limiti topilsin.

Yechish. Funksiya $0xy$ tekislikning $0(o,o)$ nuqtasidan farqli barcha nuqtalarida aniqlangan. $x^2 \le x^2+y^2$, $y^2 \le x^2+y^2$ tengsizliklardan $x^3 \le (x^2+y^2)^{\frac{3}{2}}$, $y^3 \le (x^2+y^2)^{\frac{3}{2}}$ va bulardan $x^3 + y^3 \le 2(x^2+y^2)^{\frac{3}{2}}$ tengsizlikka ega bo'lamiz.

Demak

$$|f(x;y)| \le \frac{2(x^2+y^2)^{\frac{3}{2}}}{x^2+y^2} = 2\sqrt{x^2+y^2}.$$

Istalgan $\varepsilon > 0$ son uchun $\delta = \frac{\varepsilon}{2}$ deb olinsa $\rho(P;0) = \sqrt{x^2+y^2} < \delta$ tengsizlikni qanoatlantiruvchi $0(o,o)$ nuqtadan farqli barcha $P(x;y)$ nuqtalar uchun

$$|f(x;y) - 0| \le 2\sqrt{x^2+y^2} < 2\delta = 2\cdot\frac{\varepsilon}{2} = \varepsilon$$

tengsizlik bajariladi. Bu tengsizlik $f(x;y)$ funksiya $0(o,o)$ nuqtada 0 ga teng limitga ega ekanligidan dalolat beradi.

3-ta'rif. Agar ixtiyorni $N>0$ son uchun shunday $\delta>0$ son topilsaki

$$\rho(P;P_0) = \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$$

tengsizlikni qanoatlantiruvchi $P_0(x_0;y_0)$ nuqtadan farqli barcha $P(x;y)$ nuqtalar uchun

$$|f(x;y)| > N$$

tengsizlik bajarilsa $f(x;y)$ funksiya $P(x;y) \to P_0(x_0;y_0)$ da cheksizlikka intiladi deyiladi va

$$\lim_{\substack{x\to x_0\\ y\to y_0}} f(x;y) = \infty$$

kabi yoziladi.

$f(x,y)$ funksiyaning $x,y\to\infty$ dagi limiti $\lim_{\substack{x\to\infty\\ y\to\infty}} f(x;y) = A$ haqida so'z yuritish ham mumkin. Bu tenglikni A chekli son bo'lganda istalgan $\varepsilon > 0$ son uchun shunday $N>0$ son topildiki x,y ning $|x| > N$, $|y| > N$ tengsizliklarni qanoatlantiruvchi barcha qiymatlarida $f(x,y)$ funksiya aniqlangan va $|f(x,y) - A| < \varepsilon$ tengsizlik bajariladi deb tushunish kerak.

4-ta'rif. Agar $\lim\limits_{P\to P_0} f(P)=0$ bo'lsa, u holda $z=f(P)$ funksiya P_0 nuqtada ($P\to P_0$ da) cheksiz kichik funksiya deyiladi.

Agar $z=f(P)$ funksiya P_0 nuqtada A ga teng limitga ega bo'lsa, u holda $\alpha(P)=f(P)-A$ funksiya P_0 nuqtada cheksiz kichik funksiya bo'ladi. Shunday qilib P_0 nuqtada chekli A limitga ega bo'lgan $z=f(P)$ funksiyani shu A son bilan P_0 nuqtada cheksiz kichik funksiya $\alpha(P)$ ning yig'indisi ko'rinishda tasvirlash mumkin ekan:

$$f(P)=A+\alpha(P), \quad \text{bunda} \quad \lim_{P\to P_0}\alpha(P)=0.$$

Bir o'zgaruvchi funksiyasining limiti haqidagi barcha asosiy teoremalar bir necha o'zgaruvchining funksiyasi uchun ham o'rinli bo'ladi.

Jumladan

$$\lim_{\substack{x\to x_0\\ y\to y_0}}[f(x;y)\pm\varphi(x;y)]=\lim_{\substack{x\to x_0\\ y\to y_0}}f(x;y)\pm\lim_{\substack{x\to x_0\\ y\to y_0}}\varphi(x;y),$$

$$\lim_{\substack{x\to x_0\\ y\to y_0}}(f(x;y)\cdot\varphi(x;y))=\lim_{\substack{x\to x_0\\ y\to y_0}}f(x;y)\cdot\lim_{\substack{x\to x_0\\ y\to y_0}}\varphi(x;y),$$

$$\lim_{\substack{x\to x_0\\ y\to y_0}}\frac{f(x;y)}{\varphi(x;y)}=\frac{\lim\limits_{\substack{x\to x_0\\ y\to y_0}}f(x;y)}{\lim\limits_{\substack{x\to x_0\\ y\to y_0}}\varphi(x;y)} \qquad \Big(\lim_{\substack{x\to x_0\\ y\to y_0}}\varphi(x;y)\neq 0\Big)$$

tengliklar o'rinli. Bu yerdagi x_0, y_0 o'rnida ∞ bo'lishi ham mumkin.

Limitga ega bo'lmagan funksiyaga misollar keltiramiz.

2-misol. $z=\dfrac{x-y}{x+y}$ funksiyaning $O(o;o)$ nuqtada limitga ega emasligi ko'rsatilsin.

Yechish. Bu funksiya $x+y=o$ to'g'ri chiziq nuqtalaridan tashqari Oxy tekisligining barcha nuqtalarida aniqlangan. Uning $(0,0)$nuqtada limitga ega emasligini ko'rsatish uchun $P(x,y)$ nuqta $P_o(0,0)$ nuqtaga ikki xil yo'nalish bo'yicha intilgan hollarni kuzatamiz. P nuqta Oy o'q bo'ylab $P_0(0,0)$ nuqtaga yaqinlashsa $x=0$ bo'lganligi sababli

$$\lim_{p\to p_0}\frac{x-y}{x+y}=\lim_{y\to 0}\frac{-y}{y}=-1$$

bo'ladi. P nuqta $0x$ o'q bo'ylab $P_0(0,0)$nuqtaga intilganda $y=0$ bo'lganligi sababli

$$\lim_{p\to p_0}\frac{x-y}{x+y}=\lim_{x\to 0}\frac{x}{x}=1$$

bo'ladi. Shunday qilib $P(x,y)$ nuqta P_o $(0,0)$ nuqtaga ikki xil yo'nalish bo'yicha yaqinlashganda funksiya ikki xil limitga ega bo'ldi. Bu berilgan funksiya $P_o(0,0)$ nuqtada limitga ega emas degan so'z.

3-misol. $f(x,y)=\dfrac{x^2y}{x^4+y^2}$ $(x^4+y^2\neq 0)$ funksiyaning $0(0,0)$ nuqtadagi limiti topilsin.

Yechish. Koordinatalar boshidan o'tuvchi istalgan $y=kx$ to'g'ri chiziqda funksiya

$$f(x,y)=f(x;kx)=\frac{kx^3}{x^4+k^2x^2}=\frac{kx}{x^2+k^2}$$

ko'rinishiga ega bo'lib bu funksiya $x\to 0$ da 0 ga teng limitga ega. Demak $P(x,y)$ nuqta istalgan $y=kx$ to'g'ri chiziq bo'ylab koordinatalar boshiga yaqinlashganda funksiya $0(0,0)$ nuqtada nolga teng limitga ega. Bundan funksiya koordinatalar boshida limitga ega ekanligi kelib chiqmaydi. Masalan $P(x,y)$ nuqta koordinatalar boshiga $y=x^2$ parabola bo'ylab yaqinlashganda

$$f(x,y)=f(x,x^2)=\frac{x^2\cdot x^2}{x^4+(x^2)^2}=\frac{1}{2}$$

bo'ladi. Shunday qilib $P(x,y)$ nuqta $0(0,0)$ nuqtaga har xil yo'nalishlar bo'yicha yaqinlashganda funksiya ikki xil limitga ega bo'ladi. Bu berilgan funksiya $0(0,0)$ nuqtada limitga ega emasligini anglatadi.

47.2. Bir necha o‘zgaruvchi funksiyasining uzluksizligi

$z=f(x,y)$ funksiya $P_o(x_o,y_o)$ nuqtada va uning biror atrofida aniqlangan bo‘lsin.

5-ta‘rif. Agar $\lim\limits_{P\to P_o} f(P)=f(P_o)$ bo‘lsa, u holda $z=f(x,y)$ funksiya $P_o(x_o\ ,y_o)$ nuqtada **uzluksiz** deb ataladi.

Bu ta‘rif quyidagi ta‘rifga teng kuchlidir.

6-ta‘rif. Agar ixtiyoriy $\varepsilon>o$ son uchun shunday $\delta>o$ son topilsaki,

$$\rho(P,P_o)<\delta$$

tengsizlikni qanoatlantiruvchi barcha $P(x,y)$ nuqtalar uchun

$$|f(x,y)-f(x_o,y_o)|<\varepsilon \ (\text{yoki } |f(P)-f(P_o)|<\varepsilon)$$

tengsizlik bajarilsa, u holda $z=f(x,y)$ funksiya $P_o(x_o\ ,y_o)$ nuqtada uzluksiz deyiladi.

Uzluksizlikning 5-ta‘rifga teng kuchli yana bir ta‘rifini keltiramiz. Buning uchun $\lim\limits_{P\to P_o} f(P)=f(P_o)$ tenglikni unga teng kuchli

$$\lim_{P\to P_o}[f(P)-f(P_o)]=0 \quad \text{yoki} \quad \lim_{\substack{x\to x_o\\ y\to y_o}}[f(x,y)=f(x_o,y_o)]=0$$

tenglik bilan almashtiramiz.

$$x-x_o=\Delta x,\quad y-y_o=\Delta y,\quad f(P)-f(P_o)=f(x,y)-f(x_o,y_o)=\Delta z$$

belgilashlarni kiritamiz.

Δz ifoda $z=f(x,y)$ funksiyaning $P_o(x_o\ ,y_o)$ nuqtadagi **to‘liq orttirmasi** deb ataladi. Kiritilgan belgilashlarga asoslanib

$$\Delta x=x_o+\Delta x,\qquad y=y_o+\Delta y,\quad \Delta z=f(x_o+\Delta x,\ y_o+\Delta y)-f(x_o,y_o)$$

tengliklarni hosil qilamiz.

7-ta‘rif. Agar $z=f(x,y)=f(P)$ funksiya $P_o(x_o\ ,y_o)$ nuqtada va uning biror atrofida aniqlangan bo‘lsa, hamda agar argumentning Δx va Δy cheksiz kichik orttirmalariga funksiyaning Δz cheksiz kichik to‘liq orttirmasi mos kelsa, ya‘ni

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \Delta z = 0$$

bo'lsa, u holda bu funksiya $P_o(x_o, y_o)$ nuqtada uzluksiz deb ataladi.

Biror to'plamning har bir nuqtasida uzluksiz funksiya shu to'plamda uzluksiz deb ataladi.

Agar biror $P_o(x_o, y_o)$ nuqtada uzluksizlik sharti $\lim_{P\to P_o} f(P) = f(P_o)$ bajarilmasa, u holda bu nuqta $z=f(x,y)$ funksiyaning **uzilish nuqtasi** deyiladi. Uzluksizlik sharti quyidagi hollarda bajarilmasligi mumkin.

1) $z=f(x,y)$ funksiya $P_o(x_o, y_o)$ nuqtaning biror atrofida aniqlangan, lekin $P_o(x_o, y_o)$ nuqtaning o'zida aniqlanmagan;

2) $z=f(x,y)$ funksiya $P_o(x_o, y_o)$ nuqtada va uning biror atrofida aniqlangan, lekin

$$\lim_{\substack{x\to x_o\\ y\to y_o}} f(x,y)$$

mavjud emas.

3) $z=f(x,y)$ funksiya $P_o(x_o, y_o)$ nuqtada va uning biror atrofida aniqlangan va $\lim_{\substack{x\to x_o\\ y\to y_o}} f(x,y)$ limit ham mavjud , lekin

$$\lim_{\substack{x\to x_o\\ y\to y_o}} f(x,y) \neq f(x_o, y_o).$$

4-misol. $z=x^2+y^2$ funksiyaning Oxy tekislikning istalgan nuqtasida uzluksizligi ko'rsatilsin.

Yechish. $\Delta z=[(x+\Delta x)^2+(y+\Delta y)^2]-(x^2+y^2)=[(x^2+2x\cdot\Delta x+\Delta x^2+$ $+y^2+2y\cdot\Delta y+\Delta y^2)]-(x^2+y^2)= 2x\cdot\Delta x+ 2y\cdot\Delta y+\Delta x^2+\Delta y^2$, demak $\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} \Delta z = 0$, va $z=x^2+y^2$ funksiya Oxy tekislikning ixtiyoriy $P(x,y)$ nuqtasida uzluksiz ekan.

5-misol. Ushbu $z = \dfrac{1}{x-y}$ funksiyaning uzilish nuqtalarini toping.

Yechish. Funksiya Oxy tekislikning koordinatalari $x-y=0$ $(x=y)$ tenglamani qanoatlantiruvchi nuqtalardan tashqari barcha nuqtalarida aniqlangan va uzluksiz. $y=x$ to'g'ri chiziq birinchi va uchinchi koordinata burchaklarining bissektrisasidir. Ana shu bissektrisaning har bir nuqtasi berilgan funksiyaning uzilish nuqtasi bo‘ladi. Shunday qilib uzilish nuqtalari funksiyaning uzilish to‘g‘ri chizig‘ini hosil qilar ekan.

Agar $z = f(x;y) = \begin{cases} 1 & y \geq 0 \text{ bo'lganda,} \\ 0 & y < 0 \text{ bo'lganda} \end{cases}$ funksiyani qarasak bu funksiya Oxy tekisligining Ox o‘qda yotmagan istalgan nuqtasida uzluksiz Ox o‘q funksiyaning uzilish chizig‘i bo‘ladi.

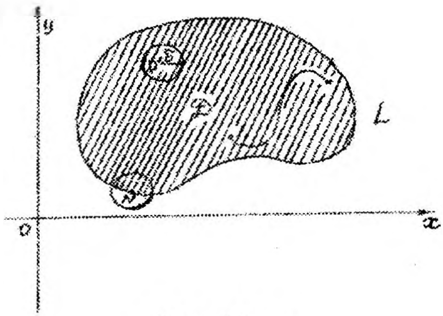
Ikki o‘zgaruvchining uzluksiz funksiyasi bir o‘zgaruvchining uzluksiz funksiyasi ega bo‘lgan barcha asosiy xossalarga ega bo‘ladi. Jumladan $(x_0;y_0)$ nuqtada uzluksiz $f(x;y)$, $\varphi(x;y)$ funksiyalarning yig‘indisi, ayirmasi, ko‘paytmasi va mahraji $\varphi(x_0;y_0) \neq 0$ bo‘lganda $f(x;y):\varphi(x;y)$ bo‘linmasi ham $(x_0;y_0)$ nuqtada uzluksiz bo‘ladi.

47.3. Yopiq sohada uzluksiz funksiyaning xossalari

Bir o‘zgaruvchining funksiyasi o‘rganilganda *[a;b]* kesmada uzluksiz *y=f(x)* funksiya shu kesmada chegaralanganligini, o‘zining eng katta qiymati *M* va eng kichik qiymati *m* ga erishishini, hamda *(m,M)* intervaldagi barcha qiymatlarni qabul qilishini bilamiz. Ikki va undan ortiq o‘zgaruvchilarning funksiyalari ham shu xossalarga o‘xshash xossalarga ega. Ularni keltirishdan oldin bizga kerak bo‘ladigan ba‘zi-bir tushunchalar bilan tanishamiz.

8-ta‘rif. Agar tekislik nuqtalarining *D* to‘plamiga tegishli ixtiyoriy ikki nuqtani shu to‘plam nuqtalaridan tashkil topgan uzluksiz chiziq bilan tutashtirish mumkin bo‘lsa, u holda *D* to‘plam **bog‘lamli to‘plam** deb ataladi.

9-ta‘rif. Agar *D* to‘plamning *P* nuqtasi shu to‘plamning nuqtalaridan tashkil topgan biror δ - atrofi bilan shu to‘plamga tegishli bo‘lsa, u holda u *D* to‘plamning **ichki nuqtasi** deyiladi (212-chizma).



212-chizma.

10-ta'rif. Faqat ichki nuqtalaridan tashkil topgan D to'plam **ochiq to'plam** deyiladi.

Uchburchak, aylana, ellipslarning ichida yotgan tekislik nuqtalari to'plami ochiq to'plamga misol bo'la oladi.

11-ta'rif. Bog'lamli ochiq to'plam **ochiq soha** yoki qisqacha **soha** deyiladi.

Masalan Oxy tekislikning ixtiyoriy yopiq egri chiziq bilan chegaralangan qismi sohaga misol bo'la oladi.

12-ta'rif. Agar N nuqtaning ixtiyoriy δ – atrofida berilgan D to'plamga tegishli bo'lgan nuqtalar ham, bu to'plamga tegishli bo'lmagan nuqtalar ham mavjud bo'lsa, u holda N nuqta D to'plamning **chegara nuqtasi** deb ataladi (212-chizma). To'plamning barcha chegara nuqtalari to'plami uning **chegarasi** deyiladi (212-chizmadagi L chiziq).

Masalan doiraning ichida yotuvchi nuqtalardan tashkil topgan soha uchun aylana chegara bo'ladi.

13-ta'rif. Soha va uning chegarasidan tashkil topgan nuqtalar to'plami **yopiq soha** deb ataladi.

14-ta'rif. D to'plamni o'z ichiga oluvchi markazi koordinatalar boshida bo'lgan doira mavjud bo'lsa, u holda D to'plam **chegaralangan to'plam** deb ataladi.

Kesma va uchburchak- chegaralangan to'plam.

To'g'ri chiziq chegaralangan to'plam bo'la olmaydi.

Ikki o'zgaruvchining funksiyasi aniqlangan yopiq chegaralangan soha bir o'zgaruvchi funksiyasi aniqlangan kesma vazifasini o'taydi.

Endi ikki o'zgaruvchining uzluksiz funksiyasini asosiy xossalarini keltiramiz.

1. Agar $z=f(P)$ funksiya chegaralagan yopiq D sohada uzluksiz bo'lsa, u holda u shu sohada chegaralangan ya'ni shunday $k>0$ o'zgarmas son mavjudki, D sohaning barcha P nuqtalari uchun

$$|f(P)|\leq k$$

tengsizlik o'rinli bo'ladi.

2. Agar $z=f(P)$ funksiya chegaralangan yopiq D sohada uzluksiz bo'lsa, u holda u shu sohada o'zining eng katta qiymati M va eng kichik qiymati m ga erishadi, ya'ni D sohada shunday P_1 va P_2 nuqtalar mavjudki

$$f(P_1)=M,\ f(P_2)=m$$

tengliklar o'rinli bo'ladi.

3. Agar $z=f(P)$ funksiya chegaralangan yopiq D sohada uzluksiz bo'lsa, u holda u shu sohada o'zining eng katta qiymati M va eng kichik qiymati m orasidagi barcha oraliq qiymatlarini qabul qiladi, ya'ni (m,M) intervaldan olingan istalgan λ uchun D sohada kamida bitta N nuqta mavjud bo'lib

$$f(N)=\lambda$$

tenglik bajariladi.

Bundan, xususiy holda D sohaning P_1 va P_2 nuqtalarida funksiya har xil ishorali qiymatlarni qabul qilganda sohada $f(P_0)=0$ shartni qanoatlantiruvchi P_0 nuqtaning mavjudligi kelib chiqadi.

Uch va undan ortiq o'zgaruvchining funksiyasi uchun ham uzluksizlik tushunchasi xuddi ikki o'zgaruvchining funksiyasiniki singari ta'riflanadi va keltirilgan xossalar ular uchun ham o'z kuchini saqlaydi.

O'z-o'zini tekshirish uchun savollar

1. Tekislikda va fazoda nuqtaning $\delta-$ atrofi nima?
2. $z=f(x,y)$ funksiyaning $P_o(x_o,y_o)$ nuqtadagi limitini ta'riflang.
3. Cheksiz kichik funksiya nima?
4. $z=f(x,y)$ funksiyaning $P_o(x_o,y_o)$ nuqtada uzluksizligini ta'riflang.
5. Funksiyaning uzilish nuqtasi nima?
6. Bog'lamli to'plamni ta'riflang.
7. To'plamning ichki va chegara nuqtalarini tariflang.
8. Ochiq to'plamni ta'riflang va unga misollar keltiring.
9. Soha deb nimaga aytiladi?
10. Yopiq soha deb nimaga aytiladi?
11. To'plam qachon chegaralangan deyiladi?
12. Chegaralangan yopiq sohada uzluksiz funksiya qanday xossalarga ega.

Mustaqil yechish uchun mashqlar

Quyidagi limitlar hisoblansin.

1. $\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{y}{x-y}$. Javob: mavjud emas.

2. $\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{\sqrt{xy+9}-3}{xy}$. Javob: $\dfrac{1}{6}$.

3. $\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{\sin(x+y)}{x+y}$. Javob: *1*.

4. $\lim\limits_{\substack{x\to 0\\ y\to 0}}\dfrac{\sin(x^2y)}{xy}$. Javob: *0*

5. $z=\dfrac{5}{x+y}$ funksiyaning uzilish nuqtalari topilsin.

6. $z=\sqrt{1-\dfrac{x^2}{4}-y^2}$ funksiyaning uzluksizlik sohasi topilsin.

7. $z = \dfrac{x+y}{(x-1)^2+(y+2)^2}$ funksiyaning uzilish nuqtasi topilsin.

8. $z = f(x,y) = \begin{cases} 2, & \text{agar} \quad xy > 0 \quad \text{bo'lsa,} \\ o, & \text{agar} \quad xy = 0 \quad \text{bo'lsa,} \\ -2, & \text{agar} \quad xy < 0 \quad \text{bo'lsa} \end{cases}$ funksiyaning grafigi yasalsin va uzilish chizig'i ko'rsatilsin.

9. $u = \sqrt{x^2+y^2+z^2-4}$ funksiyaning uzluksizlik sohasi topilsin.

48. BIR NECHA O'ZGARUVCHI FUNKSIYALARINING XUSUSIY HOSILALARI VA DIFFERENSIALLANUVCHILIGI

48.1. Bir necha o'zgaruvchi funksiyasining xususiy hosilalari

$z=f(x,y)=f(P)$ funksiya $P(x,y)$ nuqtaning biror atrofida aniqlangan bo'lsin. x o'zgaruvchiga $P(x,y)$ nuqtada Δx orttirma beramiz, y o'zgaruvchini esa o'zgarishsiz qoldiramiz. Natijada $P_1(x+\Delta x,\ y)$ nuqtani hosil qilamizki u ham $P(x,y)$ nuqtaning ko'rsatilgan atrofidan chiqib ketmasin. U holda funksiya olgan $\Delta_x z=f(P_1)-f(P)$ yoki $\Delta_x z=f(x+\Delta x,\ y)-f(x,y)$ orttirma funksiyaning $P(x,y)$ nuqtadagi x **o'zgaruvchi bo'yicha xususiy orttirmasi** deb ataladi.

Funksiyaning $P(x,y)$ nuqtadagi y **o'zgaruvchi bo'yicha** xususiy orttirmasi ham shunga o'xshash aniqlanadi:

$$\Delta_y z=f(x,y+\Delta y)-f(x,y).$$

Nihoyat x va y o'zgaruvchilar mos ravishda Δx va Δy orttirmalar olganda funksiya olgan $\Delta z=f(x+\Delta x,\ y+\Delta y)-f(x,y)$ orttirmasi $z=f(x,y)$ funksiyaning $P(x,y)$ nuqtadagi **to'la orttirmasi** deb ataladi.

1-ta'rif. Agar

$$\lim_{\Delta x\to 0}\frac{\Delta_x z}{\Delta x}\left(\lim_{\Delta y\to 0}\frac{\Delta_y z}{\Delta y}\right)$$

limit mavjud bo'lsa, u holda bu limit $z=f(x,y)$ funksiyaning $P(x,y)$ nuqtadagi x (y) **o'zgaruvchi bo'yicha xususiy hosilasi** deb ataladi va

$$z'_x,\ f'_x,\ \frac{\partial z}{\partial x},\ \frac{\partial f}{\partial x}\ (z'_y,\ f'_y,\ \frac{\partial z}{\partial y},\ \frac{\partial f}{\partial y})$$

simvollarning biri orqali belgilanadi.

Ta'rifdan ko'rinib turibdiki ikki o'zgaruvchi funksiyasining x (y) o'zgaruvchi bo'yicha xususiy hosilasi y (x) ni o'zgarmas sanab undan bir o'zgaruvchi funksiyasidan $x(y)$ bo'yicha olingan oddiy hosiladan iborat ekan. Shuning uchun xususiy hosilani topish bir o'zgaruvchining funksiyasini hosilasini topish qoidasi va formulasi asosida amalga oshiriladi.

1-misol. $z=x^3\cos y$ funksiyaning xususiy hosilalari topilsin.

Yechish.

$$z'_x=(x^3\cos y)'_x=(x^3)'_x\cos y=3x^2\cos y,$$

$$z'_y=(x^3\cos y)'_y=x^3(\cos y)'_y=-x^3\sin y,$$

chunki z'_x ni hisoblashda $\cos y$ o'zgarmas sanalib hosila belgisidan chiqariladi, z'_y ni hisoblashda esa x^3 o'zgarmas sanalib hosila belgisidan chiqarildi.

2-misol. $z=\ln(x^2+y^2)$, $\frac{\partial z}{\partial x}-?\frac{\partial z}{\partial y}-?$

$$\frac{\partial z}{\partial x}=\left(\ln(x^2+y^2)\right)'_x=\frac{(x^2+y^2)'_x}{x^2+y^2}=$$

$$\frac{(x^2)'_x+(y^2)'_x}{x^2+y^2}=\frac{2x+0}{x^2+y^2}=\frac{2x}{x^2+y^2},$$

$$\frac{\partial z}{\partial y}=\frac{2y}{x^2+y^2}.$$

Bunda $(\ln u)'=\frac{u'}{u}$ formuladan foydalanildi.

3-misol. $z=\sin(x^3y)$, $\frac{\partial z}{\partial x}-?\frac{\partial z}{\partial y}-?$

$$\frac{\partial z}{\partial x}=\left(\sin(x^3y)\right)'_x=\cos(x^3y)\cdot(x^3y)'_x=\cos(x^3y)\cdot(x^3)'_x\cdot y=3x^2y\cos(x^3y);$$

$$\frac{\partial z}{\partial y}=\left(\sin(x^3y)\right)'_y=\cos(x^3y)\cdot(x^3y)'_y=\cos(x^3y)\cdot x^3y'_y=x^3\cos(x^3y).$$

Bu yerda $(\sin u)'=\cos u\cdot u'$ va $(x^n)'=nx^{n-1}$ formulalardan foydalanildi.

4-misol. $z = \operatorname{arctg}\frac{x}{y}, \frac{\partial z}{\partial x} - ? \frac{\partial z}{\partial y} - ?$

$$\frac{\partial z}{\partial x} = \left(\operatorname{arctg}\frac{x}{y}\right)'_x = \frac{\left(\frac{x}{y}\right)'_x}{1+\left(\frac{x}{y}\right)^2} = \frac{\frac{1}{y}}{1+\frac{x^2}{y^2}} = \frac{y}{y^2+x^2};$$

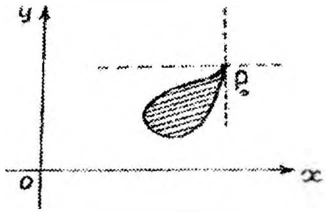
$$\frac{\partial z}{\partial y} = \left(\operatorname{arctg}\frac{x}{y}\right)'_y = \frac{\left(\frac{x}{y}\right)'_y}{1+\left(\frac{x}{y}\right)^2} = \frac{-\frac{x}{y^2}}{1+\frac{x^2}{y^2}} = -\frac{x}{y^2+x^2}.$$

Bu yerda $(\operatorname{arctg} u)' = \frac{u'}{1+u^2}$ formuladan foydalanildi.

5-misol. $z=x^y$, $\frac{\partial z}{\partial x} - ? \frac{\partial z}{\partial y} - ?$

$$\frac{\partial z}{\partial x} = (x^y)'_x = yx^{y-1}, \frac{\partial z}{\partial y} = (x^y)'_y = x^y \ln x.$$

Biz bu yerda $(x^\alpha)' = \alpha x^{\alpha-1} va (a^x)' = a^x \ln a$ formuladan foydalandik. Ikki o'zgaruvchi funksiyasining xususiy hosilasini keltirilgan ta'rifi funksiyaning aniqlanish sohasini ichki nuqtalari uchun to'g'ri keladi. Agar $P(x,y)$ nuqta funksiyani aniqlanish sohasini chegara nuqtasi bo'lsa, u holda $\Delta_x z$ ($\Delta_y z$) xususiy orttirmalar aniqlanmagan bo'lishlari ham mumkin, chunki bu holda $P_1(x+\Delta x,y)$, $P_2(x,y+\Delta y)$ nuqtalar $\Delta x \neq 0$, $\Delta y \neq 0$ orttirmalarning hech bir qiymatlarida funksiyaning aniqlanish sohasiga tegishli bo'lmasligi ham mumkin.



213-chizma.

Bu 213-chizmadagi P_0 nuqta uchun o'rinli.

Bunday holda sohaning ichki $P(x,y)$ nuqtalarida z'_x xususiy hosila hamda $\lim_{p\to p_0} z'_x(P)$ limit mavjud bo'lsa, u holda shu limit funksiyasining P_0 nuqtadagi x bo'yicha xususiy hosilasi deb qabul qilinadi, ya'ni

$$z'_x(P_0)=\lim_{p\to p_0} z'_x(P).$$

$z'_y(P_0)$ ham shunga o'xshash aniqlanadi.

Uch va undan ortiq o'zgaruvchi funksiyasining xususiy hosilasi ham shunga o'xshash ta'riflanadi va hisoblanadi. Masalan,

$$u=f(x,y,z)$$

uch o'zgaruvchining funksiyasini xususiy hosilalari

$$u'_x=\lim_{\Delta x\to 0}\frac{\Delta_x u}{\Delta x}=\lim_{\Delta x\to 0}\frac{f(x+\Delta x,y,z)-f(x,y,z)}{\Delta x},$$

$$u'_y=\lim_{\Delta y\to 0}\frac{\Delta_y u}{\Delta y}=\lim_{\Delta y\to 0}\frac{f(x,y+\Delta y,z)-f(x,y,z)}{\Delta y},$$

$$u'_z=\lim_{\Delta z\to 0}\frac{\Delta_z u}{\Delta z}=\lim_{\Delta z\to 0}\frac{f(x,y,z+\Delta z)-f(x,y.z)}{\Delta z}$$

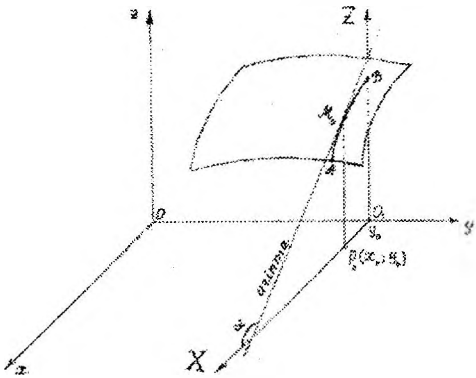
kabi aniqlanadi. Sohaning barcha nuqtalarida x (yoki y) bo'yicha xususiy hosilaga ega funksiya shu sohada x (yoki y) bo'yicha **xususiy hosilaga** ega deyiladi.

48.2. Ikki o'zgaruvchi funksiyasi xususiy hosilalarining geometrik ma'nolari

Bir o'zgaruvchining funksiyasi $y=f(x)$ qaralganda $f'(x_0)$ hosila (agar u mavjud bo'lsa) geometrik nuqtai nazardan $y=f(x)$ egri chiziqning M_0 $(x_0, f(x_0))$ nuqtasida unga o'tkazilgan urinmaning burchak koeffitsientini ya'ni urinmani Ox o'qning musbat yo'nalishi bilan tashkil etgan α burchakning tangensini anglatishi ta'kidlangan edi $(f'(x_0)=tg\,\alpha)$.

Endi $z=f(x,y)$ ikki o'zgaruvchi funksiyasi xususiy hosilalari $\frac{\partial z}{\partial x}$ va $\frac{\partial z}{\partial y}$ ning geometrik ma'nosini aniqlaymiz.

$z=f(x,y)$ tenglama 214-chizmada tasvirlangan sirtning tenglamasi bo'lsin. Oxy tekislikning $z=f(x,y)$ funksiyaning aniqlanish sohasiga tegishli $P_o(x_0, y_0)$ nuqtasini hamda sirtdagi shu nuqtaga mos $M_0\,(x_0, y_0, z_0)$ nuqtani olamiz, bunda $z_0=f(x_0, y_0)$ (214-chizma).



214-chizma.

Koordinata o'qlarini parallel ko'chirib koordinatalar boshini $O_1\,(0,y_0,0)$ nuqtaga joylashtiramiz va sirtning yangi O_1XZ koordinata tekisligi bilan kesishish chizig'i (ya'ni sirtning eski sistemadagi $y=y_0$ tekisligi bilan kesishish chizig'i) $AM_0\,B$ silliq chiziqni qaraymiz. Bu egri chiziqni O_1XZ tekislikdagi bir o'zgaruvchining funksiyasi $z=f(x,y_0)$ ning grafigi deb qarash mumkin. U holda bir o'zgaruvchi funksiyasi hosilasining geometrik ma'nosiga binoan $\frac{df(x_0,y_0)}{dx}=tg\alpha$ bo'ladi, bunda α AM_0B egri chiziqqa uning M_0 nuqtasida o'tkazilgan urinmaning O_1X yoxud ox bilan tashkil etgan burchagi.

Ikkinchi tomondan hosilaning ta'rifiga binoan

$$\frac{df(x,y_0)}{dx}\Big|_{x=x_0}=\lim_{\Delta x\to 0}\frac{f(x_0+\Delta x,y_0)-f(x_0,y_0)}{\Delta x}=\left(\frac{\partial z}{\partial x}\right)_{P_0}.$$

Bundan $\left(\frac{\partial z}{\partial x}\right)_{P_0}=tg\alpha$ kelib chiqadi.

Demak, $\frac{\partial z}{\partial x}$ xususiy hosilaning $P_0(x_0, y_0)$ nuqtadagi qiymati $\frac{\partial z(x_0,y_0)}{\partial x}$ $z=f(x,y)$ sirt bilan $y=y_0$ tekislikning kesishish chizig'ini M_0 (x_0, y_0, z_0) nuqtasida shu egri chiziqqa o'tkazilgan urinmaning ox o'q bilan tashkil etgan burchagi tangensini ifodalar ekan. Ana shu $\frac{\partial z}{\partial x}$ xususiy hosilaning geometrik ma'nosidir.

$\frac{\partial z}{\partial x}$ uchun yuritilgan mulohazalarni takrorlab $\frac{\partial z}{\partial y}$ xususiy hosilaning $P_0(x_0,y_0)$ nuqtadagi qiymati $z=f(x,y)$ sirt bilan $x=x_0$ tekislikning kesishish chizig'ini M_0 (x_0, y_0, z_0) nuqtasida shu egri chiziqqa o'tkazilgan urinmaning oy o'q bilan tashkil etgan burchak tangensini ifodalashiga iqror bo'lamiz.

48.3. Bir necha o'zgaruvchi funksiyasining differensiallanuvchiligi

$z=f(P)$ funksiya $P(x,y)$ nuqtaning biror atrofida aniqlangan bo'lsin.

2-ta'rif. Agar $z=f(P)$ funksiyaning $P(x,y)$ nuqtadagi to'la orttirmasi $\Delta z=f(x+\Delta x,y+\Delta y)-f(x,y)$, $\Delta z=A\Delta x+B\Delta y+\alpha(\Delta x,\Delta y)\Delta x+$ $+\beta(\Delta x,\Delta y)\Delta y$ (48.1)
ko'rinishda tasvirlansa, u holda $z=f(P)$ funksiya $P(x,y)$ nuqtada **differensiallanuvchi** deb ataladi, bunda A va B, $\Delta x,\Delta y$ ga bog'liq

bo'lmagan sonlar, $\alpha\ (\Delta x,\Delta y)$ va $\beta\ (\Delta x,\Delta y)$ $\Delta x\to 0,\ \Delta y\to 0$ da yoki $\rho=\sqrt{\Delta x^2+\Delta y^2}\to 0$ da cheksiz kichik funksiyalar.

Bir o'zgaruvchi funksiyasining biror nuqtada differensiallanuvchiligidan uning shu nuqtada uzluksizligi va hosilaga ega ekanligi kelib chiqar edi. Shuningdek, bir o'zgaruvchi funksiyasining biror nuqtada hosilaga ega ekanligidan uning ish nuqtada differensiallanuvchiligi kelib chiqadi. Boshqacha aytganda bir o'zgaruvchi funksiyasi uchun funksiyaning nuqtada hosilaga ega ekanligi va differensiallanuvchiligi bir narsa edi.

Shu xossa ikki o'zgaruvchining funksiyasi uchun ham saqlanadimi degan savolga javob izlaymiz.

48.1-teorema. Agar $z=f(P)$ funksiya $P(x,y)$ nuqtada differensiallanuvchi bo'lsa, u holda u shu nuqtada uzluksiz bo'ladi.

Isboti. Agar $z=f(P)$ funksiya P nuqtada differensiallanuvchi bo'lsa, u holda (48.1) munosabatdan $\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\Delta z=0$ kelib chiqadi, bu esa funksiyaning P nuqtada uzluksizligini anglatadi.

48.2-teorema. Agar $z=f(P)$ funksiya P nuqtada differensiallanuvchi bo'lsa, u holda bu funksiya shu nuqtada $f'_x(x,y), f'_y(x,y)$ xususiy hosilalarga ega va $f'_x(x,y)=A, f'_y(x,y)=B$ tengliklar o'rinlidir.

Isboti. $z=f(P)$ funksiya P nuqtada differensiallanuvchi bo'lganligi uchun *(48.1)* tenglik o'rinli. $\Delta y=0$ deb faraz qilsak undan

$$\Delta_x z=A\cdot\Delta x+\alpha\ (\Delta x,0)\Delta x \qquad (48.2)$$

tenglikka ega bo'lamiz, bunda $\lim\limits_{\Delta x\to 0}\alpha(\Delta x,0)=0$. (48.2) tenglikni har ikkala tomonini Δx ga bo'lib $\Delta x\to 0$ da limitga o'tsak

$$\lim_{\Delta x\to 0}\frac{\Delta_x z}{\Delta x}=\lim_{\Delta x\to 0}\left[A+\alpha(\Delta x,0)\right]=A$$

kelib chiqadi. Demak, $P(x,y)$ nuqtada $f'_x(x,y)$ xususiy hosila mavjud va u A ga teng.

P(x,y) nuqtada $f'_y(x,y) = B$ xususiy hosilaning mavjudligi ham shunga o'xshash isbotlanadi.

Shunday qilib, *P(x,y)* nuqtada differensiallanuvchi *z=f(P)* funksiya uchun

$$\Delta z = f'_x(x,y)\Delta x + f'_y(x,y)\Delta y + \alpha(\Delta x,\Delta y)\cdot\Delta x + \beta(\Delta x,\Delta y)\cdot\Delta y \quad (48.3)$$

tenglik o'rinli ekan.

48.1 va 48.2-teoremalarga teskari teoremalar o'rinli emas, ya'ni funksiyaning biror nuqtada uzluksizligidan yoki xususiy hosilalarga ega ekanligidan uning shu nuqtada differensiallanuvchiligi kelib chiqmaydi.

Masalan, $z = \sqrt{x^2 + y^2}$ funksiya *(0,0)* nuqtada uzluksiz, ammo u shu nuqtada xususiy hosilalarga ega emas.

Haqiqatan,

$$\frac{f(0+\Delta x,0) - f(0,0)}{\Delta x} = \frac{\sqrt{(0+\Delta x)^2} - 0}{\Delta x} = \frac{|\Delta x|}{\Delta x}.$$

Ammo $\frac{|\Delta x|}{\Delta x}$ funksiya $\Delta x \to 0$ da limitga ega emas. Demak $f'_x(0,0)$ mavjud emas; $f'_y(0,0)$ xususiy hosilaning mavjud emasligi ham shunga o'xshash ko'rsatiladi. Funksiya *(0,0)* nuqtada xususiy hosilalarga ega bo'lmaganligi uchun u shu nuqtada differentsiallanmovchi.

$$f(x,y) = \begin{cases} 0 & \text{koordinata o'qlarida,} \\ 1 & Oxy \text{ tekislikning boshqa nuqtalarida} \end{cases}$$

funksiya *(0,0)* nuqtada *x* va *y* bo'yicha $f'_x(0,0)=0$, $f'_y(0,0)=0$ xususiy hosilalarga ega. Ammo *f(x,y)* funksiya *(0,0)* nuqtada uzluksiz emas, chunki *y=x* to'g'ri chiziq bo'ylab *P(x,y)* nuqta *(0,0)* nuqtaga intilganda

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} f(x,y) = 1, \quad f(0,0) = 0 \quad \text{va} \quad \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} f(x,y) = 1 \neq f(0;0).$$

Demak, *f(x,y)* funksiya *(0,0)* nuqtada differensiallanmovchi, chunki u shu nuqtada uzilishga ega.

Agar $f(x,y)=\sqrt{|x|\cdot|y|}$ funksiyani qarasak, *u (0,0)* nuqtada uzluksiz, chunki *f(0,0)=0* va $\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}} f(x,y)=0$, shu nuqtada funksiya

$$f'_x(0,0)=\lim_{\Delta x\to 0}\frac{\sqrt{|\Delta x|\cdot 0}-0}{\Delta x}=0, \qquad f'_y(0,0)=\lim_{\Delta y\to 0}\frac{\sqrt{0\cdot|\Delta y|}}{\Delta y}=0$$

xususiy hosilalarga ega, lekin shunday bo'lsada *u (0,0)* nuqtada differensiallanuvchi emas.

Haqiqatan, funksiyaning *(0,0)* nuqtadagi to'la orttirmasi $\Delta z=\sqrt{|\Delta x|\cdot|\Delta y|}$ bo'ladi.

Agar funksiya *(0,0)* nuqtada differensiallanuvchi bo'lganda edi (48.3) ga binoan $\Delta z=0\cdot\Delta x+0\cdot\Delta y+\alpha\cdot\Delta x+\beta\Delta y$ bo'lar edi, bunda $\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\alpha=\lim\limits_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\beta=0$. U holda

$$\Delta z=\alpha\cdot\Delta x+\beta\cdot\Delta y=\frac{\alpha\cdot\Delta x+\beta\cdot\Delta y}{\rho}\cdot\rho=\gamma\cdot\rho \qquad \text{bo'ladi,}$$

bunda $\rho=\sqrt{\Delta x^2+\Delta y^2}$.

$\lim\limits_{\rho\to 0}\gamma==0$ ekanini ko'rsatamiz.

$$\lim_{\rho\to 0}\gamma=\lim_{\rho\to 0}\frac{\alpha\Delta x+\beta\Delta y}{\rho}=\lim_{\rho\to 0}\left(\alpha\cdot\frac{\Delta x}{\rho}+\beta\cdot\frac{\Delta y}{\rho}\right)$$

tenglikka egamiz.

α,β funksiyalar $\rho\to 0$ da cheksiz kichik funksiyalar, $\frac{\Delta x}{\rho}$, $\frac{\Delta y}{\rho}$ esa chegaralangan funksiyalar $\left(\frac{|\Delta x|}{\rho}\le 1,\ \frac{|\Delta y|}{\rho}\le 1\right)$ bo'lganligi uchun $\lim\limits_{\rho\to 0}\gamma=0$ bo'ladi, chunki cheksiz kichik funksiyani chegaralagan funksiyaga ko'paytmasi ham cheksiz kichik funksiya bo'ladi.

Shunday qilib berilgan funksiya *(0,0)* nuqtada differensiallanuvchi bo'lsa

$$\lim_{\rho\to0}\gamma = \lim_{\rho\to0}\frac{\Delta z}{\rho} = 0$$

bo'lar ekan.

Ammo $\Delta x = \Delta y$ bo'lganda

$$\gamma = \frac{\Delta z}{\rho} = \frac{\sqrt{|\Delta x|\cdot|\Delta y|}}{\sqrt{(\Delta x)^2+(\Delta y)^2}} = \frac{1}{\sqrt{2}}$$

va $\lim_{\rho\to0}\gamma = \frac{1}{\sqrt{2}} \neq 0$ bo'ladi. Bu qarama-qarshilik qilingan farazni noto'g'riligini ya'ni funksiya P nuqtada differensiallanuvchi emasligini tasdiqlaydi.

Shunday qilib $f(x,y) = \sqrt{|x|\cdot|y|}$ funksiya *(0,0)* nuqtada uzluksiz va xususiy hosilalarga ega bo'lishiga qaramasdan u shu nuqtada differensiallanuvchi emas ekan.

48.3-teorema. Agar $z=f(P)$ funksiya P nuqtaning biror δ -atrofida xususiy hosilalarga ega bo'lib, xususiy hosilalar P nuqtaning o'zida uzluksiz bo'lsa, u holda funksiya shu P nuqtada differensiallanuvchi bo'ladi.

Isboti. x va y o'zgaruvchilarga juda kichik Δx, Δy orttirmalar beramizki

$P_1\ (x+\Delta x,\ y+\Delta y)$ nuqta P nuqtaning δ -atrofidan chiqmasin. Funksiyaning

$$\Delta z=f(x+\Delta x,\ y+\Delta y)-f(x,y)$$

to'la orttirmasini

$$\Delta z=[f(x+\Delta x,y+\Delta y)-f(x,y+\Delta y)]+[f(x,y+\Delta y)-f(x,y)] \qquad (48.4)$$

ko'rinishda yozamiz.

$f(x+\Delta x,y+\Delta y)-f(x,y+\Delta y)$ ifodani bir o'zgaruvchi x ning funksiyasi $f(x,y+\Delta y)$ ning orttirmasi (ikkinchi argument o'zgarmas $y+\Delta y$ qiymatga ega) deb qarashimiz mumkin.

Ma'lumki, oxirlari x va $x+\Delta x$ bo'lgan kesmada uzluksiz hamda shu kesmada hosilaga ega bo'lgan $\varphi(x)$ funksiya uchun

$\varphi(x+\Delta x)-\varphi(x)=\varphi'(x+\theta\Delta x)\cdot\Delta x,\quad \theta\in(0,1)$ Lagranjning chekli orttirmalar formulasi o'rinli bo'lar edi.

Qaralayotgan holda bir o'zgaruvchi x ning funksiyasi $f(x,y+\Delta y)$ ham Lagranj teoremasining barcha shartlarini qanoatlantiradi, ya'ni bu funksiya teoremaning shartiga binoan hosila (xususiy hosila f'_x) ga ega bo'lgani uchun u uzluksiz.

Shuning uchun $f(x,y+\Delta y)$ funksiya uchun Lagranj formulasi

$f(x+\Delta x,y+\Delta y)-f(x,y+\Delta y)=f'_x(x+\theta_1\Delta x,y+\Delta y)\Delta x,\ 0<\theta_1<1$ ko'rinishga ega bo'ladi.

$f(x, y+\Delta y)-f(x,y)$ ifoda uchun ham xuddi shunday mulohaza yuritib

$$f(x, y+\Delta y)-f(x,y)= f'_y(x,y+\theta_2\Delta y)\Delta y,\ 0<\theta_2<1$$

tenglikka ega bo'lamiz. f'_x, f'_y xususiy hosilalar $P(x,y)$ nuqtada uzluksiz bo'lganligi sababli

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} f'_x(x+\theta_1\Delta x, y+\Delta y)=f'_x(x,y),$$

$$\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}} f'_y(x, y+\theta_2\Delta y)=f'_y(x,y)$$

bo'ladi. Bundan

$$f'_x(x+\theta_1\Delta x, y+\Delta y)=f'_x(x,y)+\alpha(\Delta x,\Delta y),$$
$$f'_y(x, y+\theta_2\Delta y)=f'_y(x,y)+\beta(\Delta x,\Delta y)$$

kelib chiqadi, bunda $\alpha(\Delta x, \Delta y)$, $\beta(\Delta x, \Delta y)$ funksiyalar $\Delta x\to 0$, $\Delta y\to 0$ da cheksiz kichik funksiyalar.

Shunday qilib

$$f(x+\Delta x,y+\Delta y)-f(x,y+\Delta y)= f'_x(x,y)\Delta x+\alpha(\Delta x,\Delta y)\Delta x,$$
$$f(x,y+\Delta y)-f(x,y)= f'_y(x,y)\Delta x+\beta(\Delta x,\Delta y)\Delta y$$

tenglikka ega bo'ldik. Bularni (48.4) ga qo'ysak (48.3) tenglik hosil bo'ladi. U funksiyaning P nuqtada differensiallanuvchiligini ko'rsatadi. Bundan buyon bir necha o'zgaruvchining funksiyasi biror nuqtada differensiallanuvchi deyilganda u shu nuqtaning biror atrofida aniqlangan va nuqtaning o'zida uzluksiz xususiy hosilalarga ega deb tushuniladi.

Natija. Funksiyaning xususiy hosilalarini uzluksizligidan uning o'zini ham uzluksizligi kelib chiqadi.

Keltirilgan 48.3-teorema funksiyaning differensiallanuvchiligini tekshirishda muhim o'rin tutadi. Chunki funksiyaning differensiallanuvchiligini ta'rif yordamida tekshirishdan ko'ra xususiy hosilalarning uzluksizligini tekshirish ancha oson.

Uch va undan ortiq o'zgaruvchilarning funksiyalari uchun ham differensiallanuvchilik tushunchasi xuddi ikki o'zgaruvchining funksiyasiniki kabi kiritiladi.

O'z-o'zini tekshirish uchun savollar

1. Xususiy orttirma nima?
2. To'la orttirma nima?
3. Ikki o'zgaruvchi funksiyasining xususiy hosilalarini ta'riflang?
4. Sohaning chegara nuqtalarida xususiy hosila qanday ta'riflanadi?
5. Ikki o'zgaruvchi funksiyasining xususiy hosilalarining geometrik ma'nosini ayting.
6. Ikki o'zgaruvchi funksiyasining nuqtada differensiallanuvchiligini ta'riflang.
7. Ikki o'zgaruvchi funksiyasining nuqtada differensiallanuvchiligidan uning shu nuqtada uzluksizligi kelib chiqadimi?
8. Ikki o'zgaruvchi funksiyasining nuqtada differensiallanuvchiligidan uning shu nuqtada xususiy hosilalarga ega ekanligi kelib chiqadimi?
9. Ikki o'zgaruvchi funksiyasining nuqtada uzluksizligidan uning shu nuqtada differensiallanuvchiligi kelib chiqadimi?
10. Ikki o'zgaruvchi funksiyasining nuqtada xususiy hosilalarga ega ekanligidan uning shu nuqtada differensiallanuvchiligi kelib chiqadimi?
11. Ikki o'zgaruvchi funksiyasining biror nuqtada xususiy hosilalarga ega ekanligidan uning shu nuqtada uzluksizligi kelib chiqadimi?

12. Ikki o'zgaruvchi funksiyasining biror nuqtada differensiallanuvchi bo'lishligi sharti qanday?

Mustaqil yechish uchun mashqlar

Quyidagi funksiyalarning xususiy hosilalari topilsin.

1. $z=y^2\sin^2 x$. Javob: $\frac{\partial z}{\partial x}=y^2\sin 2x, \frac{\partial z}{\partial y}=2y\sin^2 x$.

2. $z=x^{y^2}$. Javob: $\frac{\partial z}{\partial x}=y^2x^{y^2-1}, \frac{\partial z}{\partial y}=x^{y^2}\cdot 2y\cdot\ln x$.

3. $u=e^{x^2+y^2+z^2}$.

Javob: $\frac{\partial u}{\partial x}=2xe^{x^2+y^2+z^2}, \frac{\partial u}{\partial y}=2ye^{x^2+y^2+z^2}, \frac{\partial u}{\partial z}=2ze^{x^2+y^2+z^2}$.

4. $z=\operatorname{arctg}(xy)$.

Javob: $\frac{\partial z}{\partial x}=\frac{y}{1+x^2y^2}, \frac{\partial z}{\partial y}=\frac{x}{1+x^2y^2}$.

5. $z=\ln\frac{\sqrt{x^2+y^2}-x}{\sqrt{x^2+y^2}+x}$.

Javob: $\frac{\partial z}{\partial x}=-\frac{2}{x^2+y^2}, \frac{\partial z}{\partial y}=\frac{2x}{y\sqrt{x^2+y^2}}$.

6. $u=e^{\frac{x}{y}}+e^{\frac{z}{y}}$.

Javob: $\frac{\partial u}{\partial x}=\frac{1}{y}e^{\frac{x}{y}}, \frac{\partial u}{\partial y}=-\frac{x}{y^2}e^{\frac{x}{y}}-\frac{z}{y^2}e^{\frac{z}{y}}, \frac{\partial u}{\partial z}=\frac{1}{y}e^{\frac{z}{y}}$.

7. $z=\arcsin(x+y)$. Javob: $\frac{\partial z}{\partial x}=\frac{\partial z}{\partial y}=\frac{1}{\sqrt{1-(x+y)^2}}$.

49. BIR NECHA O'ZGARUVCHI FUNKSIYASINING TO'LA DIFFERENSIALI VA UNING TATBIQI

49.1. Funksiyaning to'la orttirmasi

P(x,y) nuqtada differensiallanuvchi $z=f(x,y)=f(P)$ funksiyani qaraymiz. Bu funksiyaning *P(x,y)* nuqtadagi $f_x'(x,y)$, $f_y'(x,y)$ xususiy hosilalari mavjud bo'lib

$$\Delta z = f_x'(x,y)\,\Delta x + f_y'(x,y)\,\Delta y + \alpha(\Delta x, \Delta y)\,\Delta x + \beta(\Delta x, \Delta y)\,\Delta y \quad (48.3)$$

tenglik o'rinli bo'lishi ta'kidlangan edi. Bu yerdagi $\alpha(\Delta x, \Delta y)\,\Delta x$ va $\beta(\Delta x, \Delta y)\,\Delta y$ funksiyalar *P(x,y)* va uning qaralayotgan atrofiga tegishli $P_1(x+\Delta x,\ y+\Delta y)$ nuqtalar orasidagi masofa $\rho=\sqrt{\Delta x^2+\Delta y^2}$ ga nisbatan yuqori tartibli cheksiz kichik funksiyalardir.

Haqiqatan, $\frac{\Delta x}{\rho}$ va $\frac{\Delta y}{\rho}$ chegaralangan $\left(\left|\frac{\Delta x}{\rho}\right|\le 1, \left|\frac{\Delta y}{\rho}\right|\le 1\right)$ va $\lim\limits_{\rho\to 0}\alpha(\Delta x,\Delta y)=0$, $\lim\limits_{\rho\to 0}\beta(\Delta x,\Delta y)=0$ bo'lganligi sababli cheksiz kichik funksiyani chegaralangan funksiyaga ko'paytmasi ham cheksiz kichik funksiya bo'ladi, ya'ni

$$\lim_{\rho\to 0}\frac{\alpha(\Delta x,\Delta y)\cdot\Delta x}{\rho}=0,\quad \lim_{\rho\to 0}\frac{\beta(\Delta x,\Delta y)\cdot\Delta y}{\rho}=0.$$

Shunday qilib *P(x,y)* nuqtada differensiallanuvchi $z=f(x,y)$ funksiyaning shu nuqtadagi to'la orttirmasi $\Delta z=f(x+\Delta x,\ y+\Delta y)-f(x,y)$ ni Δx, Δy orttirmalarga nisbatan chiziqli orttirmaning bosh bo'lagi $f_x'(x,y)\,\Delta x+f_y'(x,y)\,\Delta y$ bilan Δx, Δy larga nisbatan nochiziqli va $\rho=\sqrt{\Delta x^2+\Delta y^2}$ ga nisbatan yuqori tartibli cheksiz kichik funksiya $\alpha(\Delta x, \Delta y)\,\Delta x+\beta(\Delta x, \Delta y)\,\Delta y$ ning yig'indisi ko'rinishida tasvirlash mumkin ekan.

49.2. Funksiyaning to‘la differensiali

Ta‘rif. $\Delta z=f(x+\Delta x, y+\Delta y)-f(x,y)$ to‘la orttirmaning Δx, Δy larga nisbatan chiziqli bosh bo‘lagi $f_x'(x,y)\ \Delta x+\ f_y'(x,y)\ \Delta y$ differensiallanuvchi $z=f(x,y)$ funksiyaning **to‘la differensiali** yoki **differensiali** deb ataladi va dz yoki df bilan belgilanadi.

Demak, ta‘rifga binoan

$$dz=f_x'(x,y)\ \Delta x+f_y'(x,y)\ \Delta y. \qquad (49.1)$$

Ammo erkli o‘zgaruvchilarning orttirmalari ularning differensiallariga teng, ya‘ni $\Delta x=dx$, $\Delta y=dy$.

Shuning uchun (49.1) formulani

$$dz=f_x'(x,y)\ dx+f_y'(x,y)\ dy \qquad (49.2)$$

ko‘rinishida yozish ham mumkin.

To‘la differensial tushunchasidan foydalanib (48.3) tenglikni

$$\Delta z=dz+\alpha(\Delta x, \Delta y)\ \Delta x+\beta(\Delta x, \Delta y)\ \Delta y \qquad (49.3)$$

ko‘rinishida yozamiz.

Shunday qilib, differensiallanuvchi funksiyaning to‘la orttirmasi uning to‘la differensialidan $\rho=\sqrt{\Delta x^2+\Delta y^2}$ ga nisbatan yuqori tartibli cheksiz kichik miqdorga farq qilar ekan.

1- misol. $z=xy$ funktsyaining (3,4) nuqtada $\Delta x=0,1$, $\Delta y=0,2$ bo‘lganda to‘la orttirmasi va to‘la differensiali topilsin.

Yechish.

$$\Delta z=f(x+\Delta x,y+\Delta y)-f(x,y)=(x+\Delta x)(y+\Delta y)-xy=$$
$$=xy+y\Delta x+x\Delta y+\Delta x\cdot\Delta y-xy=y\,\Delta x+x\Delta y+\Delta x\cdot\Delta y,$$

$$df=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=ydx+xdy=y\Delta x+x\Delta y.$$

Demak, $\Delta z=4\cdot 0,1+3\cdot 0,2+0,1\cdot 0,2=1,02$, $dz=4\cdot 0,1+3\cdot 0,2=1$, $\Delta z-dz=$ $=1,02-1=0,02$.

2- misol. $z=\arcsin\frac{x}{y}$ funksiyaning to‘la differensiali topilsin.

Yechish. $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ xususiy hosilalarni $(\arcsin u)' = \frac{u'}{\sqrt{1-u^2}}$ formulaga asoslanib topamiz. $\frac{\partial z}{\partial x}$ ni hisoblashda y ni, $\frac{\partial z}{\partial y}$ ni hisoblashda x ni o‘zgarmas hisobalarmiz.

Shunday qilib,

$$\frac{\partial z}{\partial x} = \left(\arcsin\frac{x}{y}\right)'_x = \frac{\left(\frac{x}{y}\right)'_x}{\sqrt{1-\left(\frac{x}{y}\right)^2}} = \frac{\frac{1}{y}\cdot 1}{\sqrt{1-\frac{x^2}{y^2}}} = \frac{1}{y\sqrt{\frac{y^2-x^2}{y^2}}} = \frac{1}{y\frac{\sqrt{y^2-x^2}}{|y|}} = \frac{|y|}{y\sqrt{y^2-x^2}},$$

$$\frac{\partial z}{\partial y} = \left(\arcsin\frac{x}{y}\right)'_y = \frac{\left(\frac{x}{y}\right)'_y}{\sqrt{1-\left(\frac{x}{y}\right)^2}} = \frac{-\frac{x}{y^2}}{\sqrt{\frac{y^2-x^2}{y^2}}} = -\frac{x}{y^2\cdot\frac{\sqrt{y^2-x^2}}{|y|}} = -\frac{|y|\cdot x}{y^2\sqrt{y^2-x^2}}.$$

Demak,

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \frac{|y|\cdot dx}{y\sqrt{y^2-x^2}} - \frac{|y|\cdot x}{y^2\sqrt{y^2-x^2}}dy = \frac{|y|\cdot ydx - |y|\cdot xdy}{y^2\sqrt{y^2-x^2}}$$

$$= \frac{|y|(ydx - xdy)}{|y|^2\sqrt{y^2-x^2}} = \frac{ydx - xdy}{|y|\sqrt{y^2-x^2}}.$$

3-misol. $z = \operatorname{arctg}\frac{x+y}{1-xy}$ funksiyaning to‘la differensiali topilsin.

Yechish. Xususiy hosilalarni topishda $(\operatorname{arctg} u)' = \frac{u'}{1+u^2}$ formuladan hamda bo‘linmani hosilasini topish qiodasidan foydalanamiz.

y ni o‘zgarmas hisoblab

$$\frac{\partial z}{\partial x}=\frac{\left(\frac{x+y}{1-xy}\right)'_x}{1+\left(\frac{x+y}{1-xy}\right)^2}=\frac{\frac{1-xy+y(x+y)}{(1-xy)^2}}{\frac{(1-xy)^2+(x+y)^2}{(1-xy)^2}}=\frac{1+y^2}{1-2xy+x^2y^2+x^2+2xy+y^2}=$$

$$=\frac{1+y^2}{1+x^2+y^2(x^2+1)}=\frac{1+y^2}{(1+x^2)(1+y^2)}=\frac{1}{1+x^2}$$

tenglikka ega bo'lamiz.

Xuddi shunday

$$\frac{\partial z}{\partial y}=\frac{1}{1+y^2}$$

tenglikni hosil qilamiz.

Demak,

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy=\frac{dx}{1+x^2}+\frac{dy}{1+y^2}.$$

Keltirilgan muhokama va ta'riflar uch va undan ortiq argumentlarning funksiyalari uchun osonlikcha umumlashtiriladi.

Masalan, $u=f(x,y,z)$ uch o'zgaruvchining funksiyasi berilgan bo'lib u $P(x,y,z)$ nuqtada $\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},\frac{\partial u}{\partial z}$ uzluksiz xususiy hosilalarga ega bo'lsa, u holda $du=\frac{\partial f(x,y,z)}{\partial x}dx+\frac{\partial f(x,y,z)}{\partial y}dy+\frac{\partial f(x,y,z)}{\partial z}dz$ ifoda funksiyaning to'la orttirmasi $\Delta u=f(x+\Delta x,\ y+\Delta y,z+\Delta z)-f(x,y,z)$ ning bosh bo'lagi bo'ladi va funksiyaning to'la differensiali deb ataladi. $\Delta u-du$ ayirma $\rho=\sqrt{\Delta x^2+\Delta y^2+\Delta z^2}$ masofaga nisbatan yuqori tartibli cheksiz kichik funksiya bo'ladi.

4-misol. Uch o'zgaruvchi x,y,z ning funksiyasi $u=e^{x^3+y^3}\cos z$ funksiyaning to'la differensiali topilsin.

Yechish. Xususiy hosilalarni $(e^u)'=e^u\cdot u'$ va $(\cos x)'=-\sin x$ formuladan foydalanib topamiz. $\frac{\partial u}{\partial x}$ ni hisoblaganda y,z ni, $\frac{\partial u}{\partial y}$

hisoblaganda x,z ni va $\frac{\partial u}{\partial z}$ ni hisobalganda x,y ni o'zgarmas sanaymiz.

$$\frac{\partial u}{\partial x}=e^{x^3+y^3}3x^2\cos z, \frac{\partial u}{\partial y}=e^{x^3+y^3}3y^2\cos z, \frac{\partial u}{\partial z}=-e^{x^3+y^3}\sin z$$

xususiy hosilalar istalgan $P(x,y,z)$ nuqtada uzluksiz bo'lganligi sababli

$$du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz=e^{x^3+y^3}(3x^2\cos z dx+3y^2\cos z dy-\sin z dz).$$

5-misol. $u=(\sin x)^{yz}$ funksiyaning to'la differensiali topilsin.

Yechish. $\frac{\partial u}{\partial x}$ xususiy hosilani hisoblashda y va z ni o'zgarmas hisoblab $(u^{\alpha})'=\alpha\cdot u^{\alpha-1}u'$ formuladan foydalanamiz.

$$\frac{\partial u}{\partial x}=yz(\sin x)^{yz-1}\cdot\cos x.$$

$\frac{\partial u}{\partial y}$, $\frac{\partial u}{\partial z}$ xususiy hosilalarni hisoblashda mos ravishda x,z va x,y ni o'zgarmas sanab $(a^u)'=a^u\cdot\ln a\cdot u'$ formuladan foydalanamiz.

$$\frac{\partial u}{\partial y}=(\sin x)^{yz}\ln\sin x\cdot(yz)'_y=z(\sin x)^{yz}\ln\sin x, \quad \frac{\partial u}{\partial z}=y(\sin x)^{yz}\ln\sin x.$$

Demak, funksiyaning xususiy hosilalari uzluksiz bo'lgan barcha nuqtalarida

$$du=yz(\sin x)^{yz-1}\cos x dx+z(\sin x)^{yz}\ln\sin x dy+y(\sin x)^{yz}\ln\sin x dz$$

tenglik o'rinlidir.

Agar n o'zgaruvchining funksiyasi $u=f(x_1,x_2,...,x_n)$ biror nuqtaning atrofida barcha argumentlar bo'yicha xususiy hosilalarga ega bo'lib ular o'sha nuqtaning o'zida uzluksiz bo'lsa, u holda bu funksiyaning o'sha nuqtadagi differensiali

$$du=\frac{\partial f}{\partial x_1}dx_1+\frac{\partial f}{\partial x_2}dx_2+...+\frac{\partial f}{\partial x_n}dx_n$$

kabi topiladi.

Differensiallanuvchi funksiyaning to'la differensiali yagona bo'lishini ta'kidlab o'tamiz.

49.3. To'la differensialning taqribiy hisoblashga tatbiqi

z=f(x,y) funksiya *P(x,y)* nuqtada differensiallanuvchi bo'lsin. U holda (49.3) tenglikdan Δx, Δy orttirmalarning yetarlicha kichik qiymatlarida

$$\Delta z \approx dz$$

taqribiy tenglikka ega bo'lamiz. Bunga

$$\Delta z=f(x+\Delta x, y+\Delta y)-f(x,y), \quad dz=\frac{\partial f(x,y)}{\partial x}\Delta x+\frac{\partial f(x,y)}{\partial y}\Delta y$$

qiymatlarni qo'ysak

$$f(x+\Delta x,y+\Delta y)-f(x,y)\approx\frac{\partial f(x,y)}{\partial x}\Delta x+\frac{\partial f(x,y)}{\partial y}\Delta y$$

yoki bundan

$$f(x+\Delta x,y+\Delta y)\approx f(x,y)+\frac{\partial f(x,y)}{\partial x}\Delta x+\frac{\partial f(x,y)}{\partial y}\Delta y \qquad (49.4)$$

kelib chiqadi. Bu formuladan foydalanib *z=f(x,y)* funksiyaning hamda uning xususiy hosilalarining biror *P(x,y)* nuqtadagi aniq qiymatini bilgan holda uning shu nuqtaga yaqin $P_1(x+\Delta x,y+\Delta y)$ nuqtadagi qiymatini taqriban hisoblash mumkin.

6-misol. $\sqrt{11{,}98^2+4{,}99^2}$ kattalik to'la differensial yordamida taqriban hisoblansin.

Yechish. $f(x,y)=\sqrt{x^2+y^2}$ funksiyani qaraymiz.

$(\sqrt{u})'=\dfrac{u'}{2\sqrt{u}}$ formuladan xususiy hosilalarni topishda foydalanamiz.

$$\frac{\partial f}{\partial x}=\frac{(x^2+y^2)'_x}{2\sqrt{x^2+y^2}}=\frac{x}{\sqrt{x^2+y^2}},\ \frac{\partial f}{\partial y}=\frac{(x^2+y^2)'_y}{2\sqrt{x^2+y^2}}=\frac{y}{\sqrt{x^2+y^2}}.$$

Xususiy hosilalar $P(12;5)$ nuqtada uzluksiz bo'lganligi uchun $\sqrt{x^2+y^2}$ funksiya bu nuqtada diffrensiallanuvchi. Demak, bu funksiya uchun (49.4) formula o'rinli. Uni qo'llasak

$$\sqrt{(x+\Delta x)^2+(y+\Delta y)^2} \approx \sqrt{x^2+y^2}+\frac{x}{\sqrt{x^2+y^2}}\Delta x+\frac{y}{\sqrt{x^2+y^2}}\Delta y$$

bo'ladi. Bunga $x=12, y=5, \Delta x=-0,02, \Delta y=-0,01$ qiymatlarni qo'yib quyidagiga ega bo'lamiz:

$$\sqrt{11,98^2+4,99^2} \approx \sqrt{12^2+5^2}+\frac{12}{\sqrt{12^2+5^2}}(-0,02)+\frac{5}{\sqrt{12^2+5^2}}(-0,01)=$$

$$=13-\frac{0,24}{13}-\frac{0,05}{13}=13-\frac{0,29}{13}\approx 13-0,0223=12,9777.$$

7-misol. $\ln(\sqrt[3]{1,03}+\sqrt[4]{0,98}-1)$ taqribiy hisoblansin.

Yechish. $f(x,y)=\ln(\sqrt[3]{x}+\sqrt[4]{y}-1)$ funksiyani qaraymiz.

Agar $x=1$, $y=1$, $\Delta x=0,03$, $\Delta y=-0,02$ deb olsak (49.4) formula

$$f(1,03;0,98)\approx f(1;1)+\frac{\partial f(1;1)}{\partial x}\cdot 0,03-\frac{\partial f(1;1)}{\partial y}\cdot 0,02$$

ko'rinishni oladi.

$(\ln u)'=\frac{u'}{u}$ formulaga asoslanib topamiz:

$$\frac{\partial f}{\partial x}=\frac{(\sqrt[3]{x}+\sqrt[4]{y}-1)'_x}{\sqrt[3]{x}+\sqrt[4]{y}-1}=\frac{\frac{1}{3}x^{-\frac{2}{3}}}{\sqrt[3]{x}+\sqrt[4]{y}-1},\quad \frac{\partial f}{\partial y}=\frac{(\sqrt[3]{x}+\sqrt[4]{y}-1)'_y}{\sqrt[3]{x}+\sqrt[4]{y}-1}=\frac{\frac{1}{4}y^{-\frac{3}{4}}}{\sqrt[3]{x}+\sqrt[4]{y}-1}.$$

Funksiyani va uning xususiy hosilalarini qiymatlarini $(1;1)$ nuqtada hisoblaymiz.

$$f(1;1)=\ln(\sqrt[3]{x}+\sqrt[4]{y}-1)=\ln 1=0,$$

$$\frac{\partial f(1;1)}{\partial x}=\frac{\frac{1}{3}\cdot 1^{-\frac{2}{3}}}{\sqrt[3]{1}+\sqrt[4]{1}-1}=\frac{1}{3},\quad \frac{\partial f(1;1)}{\partial y}=\frac{1}{4}.$$

Demak,

$$f(1{,}03;0{,}98) = \ln(\sqrt[3]{1{,}03} + \sqrt[4]{0{,}98} - 1) \approx 0 + \frac{1}{3}\cdot 0{,}03 - \frac{1}{4}\cdot 0{,}02 = 0{,}005.$$

8-misol. x $x_1 = 2$ dan $x_2 = 2{,}5$ gacha, y $y_1 = 4$ dan $y_2 = 3{,}5$ gacha o'zgarganda $f(x,y) = \frac{x+3y}{y-3x}$ funksiya qanchaga o'zgaradi.

Yechish. Funksiyaning to'la orttirmasini uning to'la differensialiga almashtirib

$$\Delta f = f(x_2;y_2) - f(x_1;y_1) \approx \frac{\partial f(x_1;y_1)}{\partial x}\Delta x + \frac{\partial f(x_1;y_1)}{\partial y}\Delta y$$

yoki $\Delta x = x_2 - x_1 = 0{,}5$, $\Delta y = y_2 - y_1 = -0{,}5$ ekanini hizobga olib $\Delta f \approx \frac{\partial f(2;4)}{dx}0{,}5 + \frac{\partial f(2;4)}{\partial y}(-0{,}5)$ taqribiy tenglikni hosil qilmiz.

Bo'linmaning hosilasini topish qoidasiga asoslanib berilgan funksiyaning xususiy hosilalarini topamiz:

$$\frac{\partial f(x,y)}{\partial x} = \left(\frac{x+3y}{y-3x}\right)'_x = \frac{y-3x+3(x+3y)}{(y-3x)^2} = \frac{10y}{(y-3x)^2},$$

$$\frac{\partial f(x,y)}{\partial y} = \left(\frac{x+3y}{y-3x}\right)'_y = \frac{3(y-3x)-(x+3y)}{(y-3x)^2} = \frac{-10x}{(y-3x)^2}.$$

Xususiy hosilalarning (2;4) nuqtadagi qiymatlarini topamiz:

$$\frac{\partial f(2;4)}{\partial x} = \frac{10\cdot 4}{(4-3\cdot 2)^2} = 10, \quad \frac{\partial f(2;4)}{\partial y} = \frac{-10\cdot 2}{(4-3\cdot 2)^2} = -5.$$

Demak,

$$\Delta f \approx 10\cdot 0{,}5 + 5\cdot 0{,}5 = 7{,}5$$

ya'ni funksiya taqriban 7,5 ga o'zgarar ekan.

O'z-o'zini tekshirish uchun savollar

1. To'la orttirma necha qismdan iborat?
2. To'la orttirmaning bosh bo'lagi nima?
3. Bir necha o'zgaruvchi funksiyasi uchun to'la differensialni ta'riflang.

4. Ikki o'zgaruvchining funksiyasi uchun to'la differensialni topish formulasini yozing.

5. Uch o'zgaruvchining funksiyasi uchun to'la differensialni topish formulasini yozing.

6. Funksiyaning to'la orttirmasi uning to'la differensialidan nimaga farq qiladi?

7. Ikki o'zgaruvchining funksiyasi uchun taqribiy hisoblash formulasini yozing.

8. Uch o'zgaruvchining funksiyasi uchun taqribiy hisoblash formulasini yozing.

9. Qay vaqtda dz to'la differntsial Δz to'la orttirmaning qiymatini aniqroq ifodalaydi.

Mustaqil yechish uchun mashqlar

Quyidagi funksiyalarning to'la differensiallari topilsin.

1. $z = x^{y^3}$. Javob: $dz = y^3 \cdot x^{y^3-1}dx + x^{y^3} \cdot 3y^2 \ln x dy$.

2. $z=\mathrm{arctg}(xy)$. Javob: $dz = \dfrac{ydx + xdy}{1 + x^2y^2}$.

3. $u = e^{x^2+y^2+z^2}$. Javob: $du = 2e^{x^2+y^2+z^2}(xdx+ydy+zdz)$.

4. $z = \mathrm{arctg}\dfrac{y}{x}$. Javob: $dz = \dfrac{xdy - ydx}{x^2 + y^2}$.

5. $z=\ln(xy)$. Javob: $dz = \dfrac{dx}{x} + \dfrac{dy}{y}$.

6. $z=x^y$. Javob: $dz=yx^{y-1}dx+x^y\ln x dy$.

7. $u=x^2+y^2+xtz^3$. Javob: $du=(2x+tz^3)dx+2ydy+3xtz^2dz+xz^3dt$.

8. $f(x,y)=x^2+y^3$ bo'lsa, $f'_x(2,3)$ va $f'_y(2,3)$ hisoblansin.
Javob: $f'_x(2,3)=4$, $f'_y(2,3)=27$.

9. Agar $f(x,y)=\sqrt{x^2+y^2}$ bo'lsa, $x=1$, $y=0$, $dx=\dfrac{1}{2}, dy=\dfrac{1}{4}$ bo'lganda $df(x,y)$ hisoblansin. Javob: $\dfrac{1}{2}$.

10. $1{,}03^{3{,}001}$ kattalikni to'la differensial yordamida taqriban hisoblang. Javob: 1,09.

50. MURAKKAB FUNKSIYANING HOSILASI VA TO‘LA DIFFERENSIALI

50.1. Murakkab funksiyaning hosilasi

Bir o‘zgaruvchining funksiyasi qaralganda $u=\varphi(x)$ bo‘lganda $y=f(u)=f(\varphi(x))$ funksiya x ning murakkab funksiyasi yoki funksiyaning funksiyasi, u oraliq argument deb atalishi ta‘kidlangan edi. y ni bevosita x orqali ifodalamasdan murakkab funksiyaning hosilasini

$$\frac{du}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

formula yordamida topilishi isbotlangan edi.

Shunga o‘xshash masalani bir necha o‘zgaruvchilarning funksiyasi uchun hal etamiz.

Ikki o‘zgaruvchining funksiyasi $z=f(u,v)$ berilgan bo‘lib, u,v o‘zgaruvchilar o‘z navbatida x erkli o‘zgaruvchining funksiyalari, ya‘ni $u=u(x)$, $v=v(x)$ bo‘lsin. U holda, $z=f(u(x),v(x))$ funksiya bir o‘zgaruvchi x ning **murakkab** funksiyasi, u va v oraliq argumentlar bo‘ladi. $f(u,v)$ funksiya uzluksiz xususiy hosilalarga ega, $u(x),v(x)$ funksiyalar differensiallanuvchi deb faraz qilib, z ni bevosita x orqali ifodalamasdan hosilani topish uchun formula chiqaramiz. x o‘zgaruvchiga ixtiyoriy Δx orttirma beramiz, u holda u va v o‘zguaruvchilar mos ravishda Δu, Δv orttirmalar, z funksiya esa

$$\Delta z=f(u+\Delta u,\ v+\Delta v)-f(u,v)$$

to‘la orttirma oladi. U holda (48.3) formulaga binoan

$$\Delta z=\frac{\partial z}{\partial u}\cdot\Delta u+\frac{\partial z}{\partial v}\cdot\Delta v+\alpha(\Delta u,\Delta v)\cdot\Delta u+\beta(\Delta u,\Delta v)\cdot\Delta v \qquad (50.1.)$$

tenglikka ega bo‘lamiz, bu yerdagi $\alpha(\Delta u,\Delta v)$, $\beta(\Delta u,\Delta v)$ funksiyalar $\Delta u\to 0, \Delta v\to 0$ da cheksiz kichik funksiyalar.

(50.1.) tenglikning ikkala qismini Δx ga bo‘lamiz

$$\frac{\Delta z}{\Delta x}=\frac{\partial z}{\partial u}\cdot\frac{\Delta u}{\Delta x}+\frac{\partial z}{\partial v}\cdot\frac{\Delta v}{\Delta x}+\alpha(\Delta u,\Delta v)\cdot\frac{\Delta u}{\Delta x}+\beta(\Delta u,\Delta v)\cdot\frac{\Delta v}{\Delta x}$$

va $\Delta x\to 0$ da limitga o‘tamiz:

$$\lim_{\Delta x\to 0}\frac{\Delta z}{\Delta x}=\frac{\partial z}{\partial u}\cdot\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}+\frac{\partial z}{\partial v}\cdot\lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x}+\cdot\lim_{\Delta x\to 0}\alpha(\Delta u,\Delta v)\cdot$$
$$\cdot\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}+\lim_{\Delta x\to 0}\beta(\Delta u,\Delta v)\cdot\lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x} \qquad (50.2.)$$

bu yerdagi $\frac{\partial z}{\partial u},\frac{\partial z}{\partial v}$ xususiy hosilalar Δx ga bog'liq bo'lmaganligi sababli ularni limit belgisidan tashqariga chiqarildi. Shartga binoan $u(x), v(x)$ funksiyalar differensiallanuvchi. Shuning uchun

$$\lim_{\Delta x\to 0}\frac{\Delta u}{\Delta x}=\frac{du}{dx},\ \lim_{\Delta x\to 0}\frac{\Delta v}{\Delta x}=\frac{dv}{dx}$$

hosilalar mavjud.

$u(x)$ va $v(x)$ funksiyalar differensiallanuvchi bo'lganligi sababli ular uzluksiz, shuning uchun $\Delta x\to 0$ da $\Delta u\to 0$ va $\Delta v\to 0$, demak $\lim_{\Delta x\to 0}\alpha(\Delta u,\Delta v)=0,\ \lim_{\Delta x\to 0}\beta(\Delta u,\Delta v)=0$.

Bularni hamda $\lim_{\Delta x\to 0}\frac{\Delta z}{\Delta x}=\frac{dz}{dx}$ ekanini hisobga olib (50.2) ni quyidagi ko'rinishda yozamiz:

$$\frac{dz}{dx}=\frac{\partial z}{\partial u}\cdot\frac{du}{dx}+\frac{\partial z}{\partial v}\cdot\frac{dv}{dx}. \qquad (50.3)$$

Shunday qilib x o'zgaruvchining murakkab funksiyasi ikkita oraliq argumentning funksiyasi bo'lganda uning hosilasi (50.3) formula yordamida topilar ekan. Bu formulani ixtiyoriy sondagi argumentlarning funksiyasi uchun osonlik bilan umumlashtirish mumkin. Masalan, $z=f(u,v,t)$, $u=u(x)$, $v=v(x)$, $t=t(x)$ funksiyaning hosilasini

$$\frac{dz}{dx}=\frac{\partial z}{\partial u}\cdot\frac{du}{dx}+\frac{\partial z}{\partial v}\cdot\frac{dv}{dx}+\frac{\partial z}{\partial t}\cdot\frac{dt}{dx}$$

formuladan foydalanib topish mumkin.

1-misol. Agar $u=e^x$, $v=\cos x$ bo'lsa, $z=u^3+v^2$ murakkab funksiyaning $\frac{dz}{dx}$ hosilasini toping.

Yechish. Bu masalani ikki usul bilan hal etishimiz mumkin.

1. z ni x orqali ifodalasak $z=e^{3x}+\cos^2 x$ bitta x o'zgaruvchining murakkab funksiyasiga ega bo'lamiz.

Uni differensiallab

$$\frac{dz}{dx}=3e^{3x}+2\cos x(\cos x)'=3e^{3x}+2\cos x(-\sin x)=3e^{3x}-\sin 2x$$

ni hosil qilamiz.

2. $\frac{dz}{dx}$ hosilani z ni bevosita x orqali ifodalamasdan (50.3) formuladan foydalanib topamiz:

$$\frac{du}{dx}=e^x, \frac{dv}{dx}=-\sin x, \frac{\partial z}{\partial u}=3u^2, \frac{\partial z}{\partial v}=2v$$

bo'lgani uchun

$$\frac{dz}{dx}=3u^2\cdot e^x+2v\cdot(-\sin x)=3(e^x)^2\cdot e^x-2\cos x\sin x=3\cdot e^{3x}-\sin 2x$$

Har ikkala usul bilan yechilganda ham natija bir xil bo'ldi.

Endi qaralgan holning xususiy ko'rinishi ya'ni $y=y(x)$ bo'lganda $z=f(x,y)=f(x,y(x))$ bitta x o'zgaruvchining murakkab funksiyasi bo'lgan holni qaraymiz. Bu holda (50.3) formula

$$\frac{dz}{dx}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dx}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dx}$$

ko'rinishga ega bo'ladi. $\frac{dx}{dx}=x'=1$ bo'lgani uchun

$$\frac{dz}{dx}=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dx} \qquad (50.4)$$

bo'ladi. Bu formula $\frac{dz}{dx}$ **to'la hosilani** hisoblash formulasi deb ataladi. U ikkitadan ortiq o'zgaruvchilarning funksiyasi uchun qiyinchiliksiz umumlashtiriladi. Masalan, to'rt o'zgaruvchining funksiyasi $z=f(x,y,u.v)$ berilgan bo'lib, $y,u.v$ o'z navbatida bir o'zgaruvchi x ga bog'liq, ya'ni $y=g(x)$, $u=\varphi(x)$ $v=\psi(x)$ bo'lsa, u holda z faqat bir o'zgaruvchi x ning murakkab funksiyasi bo'ladi va $\frac{dz}{dx}$ to'la hosila

$$\frac{dz}{dx}=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dx}+\frac{\partial z}{\partial u}\cdot\frac{du}{dx}+\frac{\partial z}{\partial v}\cdot\frac{dv}{dx} \qquad (50.5)$$

formula yordamida topiladi.

2- misol. Agar $y=a\sin x$, $u=\cos x$ bo'lsa,

$$z=\frac{e^{ax}(y-u)}{a^2+1}$$

funksiyaning $\frac{dz}{dx}$ to'la hosilasi topilsin, bunda a o'zgarmas son.

Yechish. To'la hosilani

$$\frac{dz}{dx}=\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dx}+\frac{\partial z}{\partial u}\cdot\frac{du}{dx}$$

formuladan foydalanib topamiz:

$$\frac{\partial z}{\partial x}=\frac{\left(e^{ax}\right)'_x(y-u)}{a^2+1}=\frac{ae^{ax}(y-u)}{a^2+1}=\frac{ae^{ax}(a\sin x-\cos x)}{a^2+1},$$

$$\frac{\partial z}{\partial y}=\frac{e^{ax}(y-u)'_y}{a^2+1}=\frac{e^{ax}}{a^2+1},\ \frac{\partial z}{\partial u}=\frac{e^{ax}(y-u)'_u}{a^2+1}=-\frac{e^{ax}}{a^2+1},$$

$$\frac{dy}{dx}=(a\sin x)'=a\cos x,\ \frac{du}{dx}=(\cos x)'=-\sin x$$

ekanini hisobga olsak

$$\frac{dz}{dx}=\frac{ae^{ax}(a\sin x-\cos x)}{a^2+1}+\frac{ae^{ax}}{a^2+1}\cdot a\cos x+\frac{e^{ax}}{a^2+1}\cdot\sin x=$$

$$=\frac{e^{ax}(a^2\sin x-a\cos x+a\cos x+\sin x)}{a^2+1}=e^{ax}\sin x.$$

Biz shu paytgacha z bitta x erkli o'zgaruvchining murakkab funksiyasi bo'lgan hollarni qaradik. Endi z ikkita x va y erkli o'zgaruvchilarning murakkab funksiyasi bo'lgan ancha umumiy holni qaraymiz, ya'ni $z=f(u,v)$, bu yerda $u=u(x,y)$, $v=v(x,y)$.

Bu yerdagi $f(u,v), u(x,y)$ va $v(x,y)$ funksiyalar differensiallanuvchi deb faraz qilamiz. z ni bevosita x,y orqali ifodalamasdan shu funksiyaning $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ xususiy hosilalarini topish uchun formulalar chiqaramiz. Xususiy hosilani masalan, $\frac{\partial z}{\partial x}$ ni topish uchun y argument o'zgarmas hisobalanadi, u holda u va v faqat birgina x o'zgaruvchining funksiyalari bo'lib qoladi. Shuning uchun $\frac{\partial z}{\partial x}$ xususiy hosilani topish yuqorida qaralgan holga ya'ni bitta erkli o'zgaruvchining murakkab funksiyasini hosilasini topishga keladi. Faqatgina (50.3) formuladagi $\frac{dz}{dx}, \frac{du}{dx}, \frac{dv}{dx}$ hosilalarni mos ravishda $\frac{\partial z}{\partial x}, \frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}$ xususiy hosilalarga almashtirilishi lozim.

Shunday qilib, z funksiyaning xusuiy hosilalarini topish uchun

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} \tag{50.6}$$

va

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} \tag{50.7}$$

formulalarni hosil qilamiz.

Demak, ikkita erkli o'zgaruvchilarning murakkab funksiyasining biror erkli o'zgaruvchi bo'yicha xususiy hosilasini topish uchun berilgan funksiyaning oraliq argumentlar bo'yicha xusuiy hosilalarini ulardan o'sha erkli o'zgaruvchi bo'yicha xususiy hosilalarga ko'paytirib qo'shish lozim ekan.

Bu qoida ixtiyoriy sondagi erkli o'zgaruvchilarning murakkab funksiyalari uchun ixtiyoriy sondagi oraliq argumentlarda to'g'ridir.

Masalan, $z=f(u,v,t)$, bu yerda $u=u(x,y)$, $v=v(x,y)$, $t=t(x,y)$ funksiyaning xususiy hosilalari

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}+\frac{\partial z}{\partial t}\cdot\frac{\partial t}{\partial x},$$
$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}+\frac{\partial z}{\partial t}\cdot\frac{\partial t}{\partial y}$$

formulalar yordamida topiladi.

3-misol. $z=\operatorname{arctg}\frac{u}{v}$, bu yerda $u=x+y$, $v=x-y$ funksiya uchun $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=\frac{x-y}{x^2+y^2}$ munosabatning bajarilishi ko‘rsatilsin.

Yechish. (50.6) va (50.7) formulalardan foydalanib $\frac{\partial z}{\partial x}$ va $\frac{\partial z}{\partial y}$ xususiy hosilalarni topamiz:

$$\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}=\left(\operatorname{arctg}\frac{u}{v}\right)'_u\cdot(x+y)'_x+\left(\operatorname{arctg}\frac{u}{v}\right)'_v\cdot(x+y)'_x=$$
$$=\frac{\left(\frac{u}{v}\right)'_u}{1+\left(\frac{u}{v}\right)^2}\cdot 1+\frac{\left(\frac{u}{v}\right)'_v}{1+\left(\frac{u}{v}\right)^2}\cdot 1=\frac{\frac{1}{v}}{1+\frac{u^2}{v^2}}+\frac{-\frac{u}{v^2}}{1+\frac{u^2}{v^2}}=$$
$$=\frac{v}{u^2+v^2}-\frac{u}{u^2+v^2}=\frac{v-u}{u^2+v^2};$$

$$\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}=\left(\operatorname{arctg}\frac{u}{v}\right)'_u\cdot(x+y)'_y+\left(\operatorname{arctg}\frac{u}{v}\right)'_v\cdot(x-y)'_y=$$
$$=\frac{\frac{1}{v}}{1+\left(\frac{u}{v}\right)^2}\cdot 1+\frac{-\frac{u}{v^2}}{1+\left(\frac{u}{v}\right)^2}\cdot(-1)=\frac{v}{v^2+u^2}+\frac{u}{v^2+u^2}=\frac{v+u}{v^2+u^2}.$$

Demak,

$$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=\frac{v-u}{u^2+v^2}+\frac{v+u}{u^2+v^2}=\frac{2v}{u^2+v^2}=$$

$$=\frac{2(x-y)}{(x+y)^2+(x-y)^2}=\frac{2(x-y)}{2(x^2+y^2)}=\frac{x-y}{x^2+y^2}$$

Shuni isbotlash talab etilgan edi.

50.2. To'la differensial shaklining invariantliligi

Bir o'zgaruvchining funksiyasi $y=f(u)$ qaralganda bu funksiyaning differensiali u erkli o'zgaruvchi bo'lishi yoki biror x erkli o'zgaruvchining funksiyasi (oraliq argument) bo'lishidan qat'iy nazar

$$dy=f'(u)du$$

ko'rinishga ega bo'lishi, ya'ni diffenrensial shaklining o'zgarmasligi (saqlanishi) isbotlangan edi va differensialning bu xossasi differentsial shaklining invarinatligi deb atalgan edi.

Bu yerda erkli o'zgaruvchilarning funksiyasini to'la differensialini shu funksiya oraliq argumentlarning funksiyasidan iborat bo'lgandagi uning to'la differensiali bilan taqqoslaymiz.

x va y erkli o'zgaruvchilarning differensiallanuvchi $z=f(x,y)$ funksiyasining to'la differensiali

$$dz=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy \qquad (50.8)$$

formula yordamida topilishi ma'lum.

Endi $z=f(u,v)$, bu yerda $u=u(x,y),v=v(x,y)$ x va y erkli o'zgaruvchilarning murakkab funksiyasini qaraymiz. $f(u,v)$, $u(x,y)$, $v(x,y)$ funksiyalar differensiallanuvchi deb faraz qilamiz. (50.6) va (50.7) formulalar yordamida aniqlangan $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ xususiy hosilalarning qiymatlarini to'la differensialni formulasi (50.8) ga qo'yamiz:

$$dz=\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}\right)dx+\left(\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}\right)dy.$$

Qavslarni ochib qo'shiluvchilarni quyidagicha guruhlaymiz.

$$dz=\frac{\partial z}{\partial u}\left(\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy\right)+\frac{\partial z}{\partial v}\left(\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy\right)$$

Qavs ichidagi ifodalar mos ravishda *du* va *dv* to'la differentsiallarni ifodalaganligi uchun

$$dz=\frac{\partial z}{\partial u}du+\frac{\partial v}{\partial v}dv \qquad (50.9)$$

formulaga ega bo'lamiz.

(50.8) va (50.9) formulalarni taqqoslab quyidagi xulosaga kelamiz. *dz* to'la differensial argumentlar erkli o'zgaruvchi bo'lishi yoki erkli o'zgaruvchilarning funksiyalari bo'lishidan qat'iy nazar bir xil shaklni saqlaydi.

To'la differensialning bu xossasi uning shaklini **invariantligi** deb ataladi. Chiqarilgan xulosa ixtiyoriy sondagi o'zgaruvchilarning funksiyalari uchun osongina umumlashtiriladi.

4-misol. Ushbu $z=u^3v^3$, $u=x^2\sin y$, $v=x^2e^y$ murakkab funksiyaning to'la differensiali topilsin.

Yechish. (50.9) formulaga ko'ra:

$$dz=(u^3v^3)'_u\cdot du+(u^3v^3)'_v dv=3u^2v^3(u'_xdx+u'_ydy)+3u^3v^2(v'_xdx+v'_ydy)=$$

$$=3u^2v^3(2\sin y dx+x^2\cos y dy)+3u^3v^2(2xe^ydx+x^2e^ydy).$$

Bu ifodani bunday yozish ham mumkin:

$$dz=\left(3u^2v^3 2x\sin y+3u^3v^2 2xe^y\right)dx+$$

$$+\left(3u^2v^3x^2\cos y+3u^3v^2x^2e^y\right)dy=\frac{\partial z}{\partial x}dx+\frac{\partial z}{\partial y}dy$$

O'z-o'zini tekshirish uchun savollar

1. Murakkab funksiya nima?

2. Bir o'zgaruvchining murakkab funksiyasini hosilasi qanday topiladi?

3. $z=f(u,v)$, $u=u(x)$, $v=v(x)$ murakkab funksiyaning hosilasini topish formulasini yozing.

4. To'la hosila nima?

5. To'la hosilani hisoblash formulalarini yozing.

6. $z=f(u,v)$, $u=u(x,y), v=v(x,y)$ murakkkab funksiyaning xususiy hosilalari qanday topiladi?

7. Bir o'zgaruvchining murakkab funksiyasi uchun differensial shaklining invarinatligi nimani anglatadi?

8. Ikkita erkli o'zgaruvchilarning murakkab funksiyasi uchun differentsial shaklining invarinatligi nimani anglatadi?

Mustaqil yechish uchun mashqlar

1. Agar $u=\ln x$, $v=\sin x$ bo'lsa, $z=u^2+v$ murakkab funksiyaning $\frac{dz}{dx}$ hosilasi topilsin. Javob: $\frac{dz}{dx}=\frac{2\ln x}{x}+\cos x$.

2. Agar $u=x^3$, $v=\sin 3x$, $t=\mathrm{arctg}\,\frac{1}{x}$ bo'lsa, $z=e^{u-2v+3t}$ murakkab funksiyaning $\frac{dz}{dx}$ hosilasi topilsin.

Javob: $e^{x^2-2\sin 3x+3\mathrm{arctg}\frac{1}{x}}\left(3x^2-6\cos 3x-\frac{3}{x^2+1}\right)$.

3. Agar $y=x^3$ bo'lsa, $z=\ln(e^x+e^y)$ funksiyaning to'la hosilasi topilsin. Javob: $\frac{dz}{dx}=\frac{e^x+3x^2e^{x^3}}{e^x+e^{x^3}}$.

4. Agar $u=\frac{x}{y}$, $v=3x-2y$ bo'lsa, $z=u^2\ln v$ murakkab funksiyaning $\frac{\partial z}{\partial x}$ va $\frac{\partial z}{\partial y}$ xususiy hosilalari topilsin.

Javob: $\frac{\partial z}{\partial x}=\frac{2x}{y^2}\ln(3x-2y)+\frac{3x^2}{y^2(3x-2y)}$,

$$\frac{\partial z}{\partial y}=-\frac{2x^2}{y^3}\ln(3x-2y)-\frac{2x^2}{y^2(3x-2y)}.$$

5. Agar $u=e^{x+y^2}$ $v=x^2+y$ bo'lsa, $z=\ln(u^2+v)$ murakkab funksiyaning $\frac{\partial z}{\partial x}$ va $\frac{\partial z}{\partial y}$ xususiy hosilalari topilsin.

Javob: $\frac{\partial z}{\partial x}=\frac{2}{e^{2x+2y^2}+x^2+y}(e^{2x+2y^2}+x)$,

$$\frac{\partial z}{\partial y}=\frac{1}{e^{2x+2y^2}+x^2+y}(4ye^{2x+2y^2}+1).$$

6. Agar $y=\sin x$ bo'lsa, $z=x^2+\sqrt{y}$ murakkab funksiyaning to'la hosilasi topilsin. Javob: $2x+\frac{\cos x}{2\sqrt{\sin x}}$.

7. Agar $u=-\cos x$, $v=\cos x$, bo'lsa, $z=\sqrt{\frac{1+u}{1+v}}$ murakkab funksiyaning $\frac{dz}{dx}$ to'la hosilasi topilsin. Javob: $\frac{dz}{dx}=\frac{1}{2\cos^2\frac{x}{2}}$.

8. Agar $u=x^2+\sin y$, $v=\ln(x+y)$ bo'lsa, $z=u+v^2$ murakkab funksiyaning $\frac{\partial z}{\partial x}$ va $\frac{\partial z}{\partial y}$ xususiy hosilalari topilsin.

Javob: $\frac{\partial z}{\partial x}=2x+2\frac{\ln(x+y)}{x+y}$, $\frac{\partial z}{\partial y}=\cos y+2\frac{\ln(x+y)}{x+y}$.

51. OSHKORMAS FUNKSIYANING HOSILASI. SIRTGA URINMA VA NORMAL

51.1. Oshkormas funksiya

Agar x va y o'zgaruvchilarning qiymatlari

$$F(x,y)=0 \qquad (51.1)$$

tenglama bilan bog'langan bo'lsa va (a,b) intervalda aniqlangan $y=f(x)$ funksiya mavjud bo'lib, uni (51.1) tenglamaga qo'yilganda *u* x ga nisbatan ayniyatga aylansa, u holda $y=f(x)$ funksiya (51.1) tenglama yordamida oshkormas shaklda berilgan funksiya ya'ni **oshkormas funksiya** deyiladi. Bu holda (51.1) tenglama (a,b) intervalda oshkormas funksiyani aniqlaydi deymiz.

$y=f(x)$ ko'rinishda berilgan funksiyani **oshkor** shaklda berilgan funksiya deb ataladi. (51.1) tenglamani y ga nisbatan yechish natijasida oshkormas funksiya oshkor funksiyaga keltiriladi.

(51.1) ko'rinishdagi har qanday tenglama ham oshkormas funksiyani aniqlayvermaydi, masalan

$$x^4+y^4+a^4=0 \quad (a\neq 0)$$

tenglama hech qanday oshkormas funksiyani aniqlamaydi, chunki u x va y ning hech bir qiymatida bajarilmaydi.

51.2. Oshkormas funksiyaning mavjudligi haqida teorema

Qanday shartlarda $F(x,y)=0$ tenglama y ni x ning oshkormas funksiyasi sifatitida aniqlaydi degan savolga **mavjudlik teoremasi** deb ataluvchi quyidagi teorema javob beradi. Uni isbotsiz keltiramiz.

51.1-teorema. Agar $F(x,y)$ funksiya hamda uning $F'_x(x,y)$, $F'_y(x,y)$ xususiy hosilalari $P_0(x_0,y_0)$ nuqtaning biror atrofida aniqlangan va uzluksiz bo'lib, $F(x_0,y_0)=0$ $F'_y(x_0,y_0)\neq 0$ bo'lsa, u holda $F(x,y)=0$ tenglama shu atrofda yagona $y=f(x)$ oshkormas funksiyani aniqlaydi va *u* x_0 nuqtani o'z ichiga oluvchi biror intervalda differensiallanuvchi bo'ladi va $f(x_0)=y_0$.

Ikkita o'zgaruvchi x va y ning oshkormas funksiyasi z uchta o'zgaruvchi miqdorlarni bog'lovchi

$$F(x,y,z)=0$$

tenglama yordamida berilishi mumkin. Bu holda ham yuqorida keltirilgan mavjudlik teoremasiga o'xshash teorema o'rinli.

51.2-teorema. Agar $F(x,y,z)$ funksiya hamda uning $F'_x(x,y,z)$, $F'_y(x,y,z)$, $F'_z(x,y,z)$ xususiy hosilalari $P_0(x_0,y_0,z_0)$ nuqtaning biror atrofida aniqlangan va uzluksiz bo'lib, $F(x_0,y_0,z_0)=0$, $F'_z(x_0,y_0,z_0)\neq 0$ bo'lsa, u holda $F(x,y,z)=0$ tenglama shu atrofda yagona $z=f(x,y)$ oshkormas funksiyani aniqlaydi va u (x_0, y_0) nuqtani o'z ichiga oluvchi biror atrofda differensiallanuvchi bo'ladi va $f(x_0, y_0)=z_0$.

51.3. Oshkormas funksiyaning hosilasi

x ning oshkormas funksiyasi $y=f(x)$, $F(x,y)=0$ tenglama yordamida berilganda tenglamani y ga nisbatan yechmasdan shu funksiyaning hosilasini topish bilan shug'ullanamiz.

$F(x,y)$ funksiya 5.1-mavjudlik teoremasining barcha shartlarini qanoatlantirsin. Tenglamadagi y o'rniga $f(x)$ ni qo'ysak

$$F(x, f(x))=0$$

ayniyat hosil bo'ladi.

Aynan nolga teng funksiyaning hosilasi ham nolga teng bo'lgani uchun to'la hosila $\frac{dF}{dx}=0$ bo'ladi.

(50.4) formulaga binoan

$$\frac{dF}{dx}=\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\cdot\frac{dy}{dx},$$

shuning uchun

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\cdot\frac{dy}{dx}=0,$$

bundan

$$\frac{\partial F}{\partial y}\cdot\frac{dy}{dx}=-\frac{\partial F}{\partial x},\frac{dy}{dx}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}},\quad\left(y'_x=-\frac{F'_x(x,y)}{F'_y(x,y)}\right).\qquad(51.2)$$

Bir o'zgaruvchi oshkormas funksiyasining hosilasi ana shu formula yordamida topiladi.

1-misol. $x^4-2x^3+3y^2+xy-4=0$ tenglama bilan berilgan y oshkormas funksiyaning hosilasi topilsin.

Yechish. $F(x,y)= x^4-2x^3+3y^2+xy-4$ belgilashni kiritamiz va $F(x,y)$ funksiya mavjudlik teoremasi 51.1 ning barcha shartlarini qanoatlantirishini tekshiramiz. Bu funksiya Oxy tekislikning hamma nuqtalarida aniqlangan va uzluksiz. Uning xususiy hosilalari

$$F'_x(x,y)=4x^3-6x^2+y,\quad F'_y(x,y)=6y+x$$

ham Oxy tekislikda uzluksiz. Shuning uchun tekislikning $F'_y(x,y)\neq 0$ bo'ladigan barcha nuqtalarida funksiya (51.2) formula bilan aniqlanadigan

$$\frac{dy}{dx}=y'_x=-\frac{F'_x}{F'_y}=-\frac{4x^3-6x^2+y}{6y+x}$$

hosilaga ega.

Xususiy holda $P_0(2;-2)$ nuqtada hosila

$$\left.\frac{dy}{dx}\right|_{\substack{x=2\\ y=-2}}=\frac{4\cdot 2^3-6\cdot 2^2+(-2)}{6\cdot(-2)+2}=-0{,}6$$

bo'ladi.

2-misol. $x^3y^2+e^{xy^2}-y=0$ egri chiziqqa $P_0(0;1)$ nuqtada urinma va normalning tenglamalari yozilsin.

Yechish. $F(x,y)=$ $x^3y^2+e^{xy^2}-y$ belgilashni kiritsak $F(x,y)$ funksiya Oxy tekislikning hamma nuqtalarida, jumladan $P_0(0;1)$ nuqtaning atrofida ham mavjudlik teoremasi 51.1 ning barcha shartlarini qanoatlantiradi. $F(x,y)$ funksiyaning xususiy hosilalarini hamda ularning $P_0(0;1)$ nuqtadagi qiymatlarini topamiz:

$$F'_x(x,y)=3x^2y^2+y^2e^{xy^2}\,;\quad F'_y(x,y)=2x^3y+2xye^{xy^2}-1;$$

$$F'_x(0;1)=1,\quad F'_y(0;1)=-1.$$

(51.1) formuladan foydalanib urinmaning burchak koeffitsientini aniqlaymiz

$$K_{ur}=\left.\frac{dy}{dx}\right|_{\substack{x=0\\ y=2}}=-\frac{F'_x(0;1)}{F'_y(0;1)}=-\frac{1}{-1}=1.$$

Demak, normalni burchak koeffitsienti $K_n=-\dfrac{1}{K_{ur}}=-\dfrac{1}{-1}=-1$.

Berilgan nuqtadan o'tuvchi to'g'ri chiziq tenglamasi $y-y_0=\kappa(x-x_0)$ ga asoslanib urinma va normal tenglamalarini topamiz. Unga $x_0=0$, $y_0=1$, $\kappa=1$ qiymatlarni qo'ysak $y-1=1(x-0)$ yoki $x-y+1=0$ urinma tenglamasi hosil bo'ladi. Shuningdek o'sha tenglamaga $x_0=0$, $y_0=1$, $\kappa=-1$ qiymatlarni qo'ysak $y-1=-1(x-0)$ yoki $x+y-1=0$ normal tenglamasi hosil bo'ladi.

Endi ikkita x va y o'zgaruvchilarning oshkormas funksiyasi $z=F(x,y)$ ning xususiy hosilalari $\dfrac{\partial z}{\partial x},\dfrac{\partial z}{\partial y}$ ni topish uchun formulalar chiqaramiz.

Oshkormas $z=f(x,y)$ funksiya $F(x,y,z)=0$

tenglama yordamida berilgan va $F(x,y,z)$ funksiya mavjudlik teoremasi 51.2. ning barcha shartlarini qanoatlantiradi deb faraz qilib xususiy hosilalarini topamiz.

$\dfrac{\partial z}{\partial x}$ ni izlaganda y o'zgarmas hisoblanadi. Shuning uchun x ni erkli o'zgaruvchi va z ni funksiya hisoblab (51.2) formuladan foydalanishimiz mumkin.

Demak,

$$\frac{\partial z}{\partial x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}\left(z'_x=-\frac{F'_x(x,y,z)}{F'_y(x,y,z)}\right). \qquad (51.3)$$

Xuddi shunday fikrlab

$$\frac{\partial z}{\partial y}=-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}\left(z'_y=-\frac{F'_y(x,y,z)}{F'_x(x,y,z)}\right). \qquad (51.4)$$

formulaga ega bo'lamiz.

3-misol. $\sin z - xyz = 0$ tenglama bilan berilgan z oshkormas funksiyaning xususiy hosilalari topilsin.

Yechish. $F(x,y,z)=\sin z - xyz$ desak $F(x,y,z)$ funksiya $Oxyz$ fazoning barcha nuqtalarida aniqlangan va uzluksiz. Uning xususiy hosilalari

$$F'_x(x,y,z)=-yz;\ \ F'_y(x,y,z)=-xz;\ \ F'_z(x,y,z)=\cos z - xy$$

lar ham butun $Oxyz$ fazoda uzluksiz funksiyalardir. Shuning uchun $F'_z(x,y,z)=\cos z - xy \neq 0$ bo'ladigan fazoning nuqtalarida z funksiya (51.3) va (51.4) formulalar bilan hisoblanadigan z'_x va z'_y xususiy hosilalarga ega bo'ladi.

$$z'_x=-\frac{-yz}{\cos z - xy}=\frac{yz}{\cos z - xy};\ \ z'_y=-\frac{-xz}{\cos z - xy}=\frac{xz}{\cos z - xy}.$$

51.4. Sirtga urinma tekislik va normal

1-ta'rif. Sirtning M_0 nuqtasidan o'tuvchi tekislik bilan M_0 hamda sirtning ixtiyoriy M nuqtalari orqali o'tuvchi kesuvchi to'g'ri chiziq orasidagi burchak M nuqta sirt bo'ylab M_0 nuqtaga intilganda nolga intilsa tekislik sirtga M_0 nuqtada urinma tekislik deyiladi. M_0 nuqta urinish nuqtasi deyiladi.

2-ta'rif. Urinish nuqtasida urinma tekislikka perpendikulyar bo'lgan to'g'ri chiziq shu nuqtada sirtga o'tkazilgan normal deb ataladi.

Endi urinma tekislik hamda normalning tenglamaalrini tuzish bilan shug'ulanamiz.

Faraz qilaylik sirt $z=f(x,y)$ tenglama bilan berilgan bo'lib, $f(x,y)$ funksiya $P_0(x_0,y_0)$ nuqtada differensiallanuvchi bo'lsin.

Sirtga $M_0(x_0,y_0,z_0)$ bunda $z_0=f(x_0,y_0)$ nuqtada urinma tekislik tenglamasi

$$z-z_0=f'_x(x_0,y_0)(x-x_0)+f'_y(x_0,y_0)(y-y_0) \qquad (51.5)$$

ko'rinishga ega bo'lishini ko'rsatamiz. (51.5) tenglamani berilgan $M_0(x_0,y_0,z_0)$ nuqtadan o'tib $\vec{N}\{A,B,C\}$ normal vektorga ega tekislik tenglamasi $A(x-x_0)+B(y-y_0)+C(z-z_0)=0$ bilan taqqoslab u $M_0(x_0,y_0,z_0)$ nuqtadan o'tib $\vec{n}\{f'_x(P_0);\ f'_y(P_0);-1\}$ normal vektorga ega tekislik tenglamasi ekaniga amin bo'lamiz. Shu tekislik haqiqatdan sirtga $M_0(x_0,y_0,z_0)$ nuqtada urinma tekislik ekanini ko'rsatish uchun $\vec{n}$ normal vektor bilan sirtni $M_0(x_0,y_0,z_0)$ va $M(x,y,z)$ nuqtalari orqali o'tuvchi kesuvchi to'g'ri chiziqning yo'naltiruvchi vektori $\overline{M_0M}=\{x-x_0,y-y_0,z-z_0\}$ orasidagi burchak M nuqta sirt bo'ylab M_0 nuqtaga yaqinlashganda $\frac{\pi}{2}$ ga intilishini ko'rsatish kifoyadir. $\vec{n}$ va $\overrightarrow{M_0M}$ vektorlar orasidagi burchakni φ bilan belgilasak ikki vektor orasidagi burchakni topish formulasiga binoan

$$\cos\varphi=\frac{\vec{n}\cdot\overline{M_0M}}{|\vec{n}|\cdot|\overline{M_0M}|}=\frac{f'_x(P_0)(x-x_0)+f'_y(P_0)(y-y_0)-(z-z_0)}{\sqrt{f'^2_x(P_0)+f'^2_y(P_0)+1}\cdot\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}$$

bo'ladi. $x-x_0=\Delta x$, $y-y_0=\Delta y$, $z-z_0=\Delta z$ belgilashlarni kiritsak

$$\cos\varphi = \frac{f'_x(P_0)\Delta x + f'_y(P_0)\Delta y - \Delta z}{\sqrt{f'^2_x(P_0) + f'^2_y(P_0) + 1}\cdot\sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}}$$

va $f'_x(P_0)\Delta x + f'_y(P_0)\Delta y = dz$ bo'lganligi sababli

$$|cos\varphi| \le \frac{|dz - \Delta z|}{\sqrt{f'^2_x(P_0) + f'^2_y(P_0) + 1}\cdot\sqrt{\Delta x^2 + \Delta y^2}}$$

bo'ladi.

$dz - \Delta z$ ayirma $\rho = \sqrt{\Delta x^2 + \Delta y^2}$ ga nisbatan yuqori tartibli cheksiz kichik miqdor ekanligini hisobga olsak tengsizlikdan $\rho \to 0$ da $|\cos\varphi| \to 0$ kelib chiqadi. Bu esa $\lim\limits_{M \to M_0} \varphi = \frac{\pi}{2}$ ekanini anglatadi.

Demak $z = f(x,y)$ sirtga $M_0(x_0,y_0,z_0)$ nuqtada urinma tekislik tenglamasi (51.5) formula yordamida topilar ekan.

4-misol. $z=2x^2+4y^2$ sirtga $M_0(2;1;12)$ nuqtasida urinma tekislik tenglamasi topilsin.

Yechish. $f(x,y)= 2x^2+4y^2$ funksiya $0xy$ tekislikda differensiallanuvchi bo'lgani uchun u berilgan $P_0(2;1)$ nuqtada ham differentsiallanuvchi. Xususiy hosilalarni hamda ularning $P_0(2;1)$ nuqtadagi qiymatlarini hisoblaymiz.

$$f'_x(x,y) = 4x, \quad f'_y(x,y) = 8y, \quad f'_x(P_0) = f'_x(2;1) = 4\cdot 2 = 8,$$

$$f'_y(P_0) = f'_x(2;1) = 8\cdot 1 = 8.$$

Bularni hamda $x_0=2$, $y_0=1$ qiymatlarni (51.5) tenglamaga qo'yib urinma tekislikning tenglamasini hosil qilamiz:

$$z-12=8(x-2)+8(y-1); \quad 8x+8y-z=12.$$

Agar sirt tenglamasi z ga nisbatan yechilmagan

$$F(x,y,z)=0$$

ko'rinishdagi tenglama bilan berilgan bo'lsa,u holda $F(x,y,z)$ funksiya $M_0(x_0,y_0,z_0)$ nuqtaning atrofida oshkormas funksiyaning mavjudlik teoremasi 51.2 ning shartlarini qanoatlantiradi va $F(x,y,z)=0$ tenglama z ni x va y ning funksiyasi sifatida aniqlaydi, ya'ni $z=$

$f(x,y)$ bunda $z_0 = f(x_0,y_0)$ deb faraz qilib, (51.3) va (51.4) formulalar bo'yicha $P_0(x_0,y_0)$ nuqtadagi xususiy hosilalarni topamiz:

$$f'_x(x_0,y_0) = -\frac{F'_x(x_0,y_0,z_0)}{F'_z(x_0,y_0,z_0)}, \quad f'_y(x_0,y_0) = -\frac{F'_y(x_0,y_0,z_0)}{F'_z(x_0,y_0,z_0)}.$$

Xususiy hosilalarning bu qiymatlarini urinma tekislikning (51.5) tenglamaga qo'yib

$$-\frac{F''_x(M_0)}{F'_z(M_0)}(x-x_0) - \frac{F'_y(M_0)}{F'_z(M_0)}(y-y_0) - (z-z_0) = 0$$

tenglamani yoki uni $-F'_z(M_0) \neq 0$ ga ko'paytirib

$$F'_x(M_0)(x-x_0) + F'_y(M_0)(y-y_0) + F'_z(M_0)(z-z_0) = 0 \quad (51.6)$$

urinma tekislik tenglamasini hosil qilamiz.

5-misol. $x^2-4y^2+2z^2=6$ sirtga $M_0(2;2;3)$ nuqtada urinma tekislik tenglamasi topilsin.

Yechish. $F(x,y,z)=x^2-4y^2+2z^2-6$ belgilashni kiritamiz. $F(x,y,z)$ funksiyaning $M_0(2;2;3)$ nuqtadagi xususiy hosilalarini topamiz:

$$F'_x = 2x, \quad F'_y = -8y, \quad F' = 4z, \quad F'_x(M_0) = 2 \cdot 2 = 4,$$
$$F'_y(M_0) = -8 \cdot 2 = -16, \quad F'_z(M_0) = 4 \cdot 3 = 12.$$

Bularni hamda $x_0=2$, $y_0=2$, $z_0=3$ qiymatlarni (51.6) tenglikka qo'yib berilgan sirtga $M_0(2;2;3)$ nuqtada urinma tekislik tenglamasini hosil qilamiz:

$$4(x-2)-16(y-2)+12(z-3)=0; \ (x-2)-4(y-2)+3(z-3)=0; \ x-4y+3z-3=0.$$

Endi sirtga uning biror nuqtasida o'tkazilgan normal to'g'ri chiziqning tenglamasini topish bilan mashg'ul bo'lamiz.

Sirtga o'tkazilgan normalning tenglamasini keltirib chiqarish uchun tekislik bilan to'g'ri chiziqning perpendikulyarlik shartidan foydalanamiz. Bu shartga binoan normalning yo'naltiruvchi vektori sifatida urinma tekislikning normal vektorini olish mumkin.

Agar sirt $z=f(x,y)$ tenglama bilan berilgan bo'lsa, u holda urinma tenglamasi (51.5) ni berilgan $M_0(x_0,y_0,z_0)$ nuqtadan o'tib $\vec{N}=\{A;B;C\}$ normal vektorga ega tekislik tenglamasi

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0$$

bilan taqqoslab $\vec{n}=\{f'_x(x_0,y_0),f'_y(x_0,y_0),-1\}$ vektor urinma tekislikning normal vektori ekaniga amin bo'lamiz. Ana shu vektor sirtga $M_0(x_0,y_0,z_0)$ nuqtada normalning yo'naltiruvchi vektori bo'ladi.

$M_0(x_0,y_0,z_0)$ nuqtadan o'tib $\vec{S}=\{m;n;p\}$ yo'naltiruvchi vektorga ega to'g'ri chiziq tenglamasi

$$\frac{x-x_0}{m}=\frac{y-y_0}{n}=\frac{z-z_0}{p}$$

ga m,n,p o'rniga $\vec{n}$ vektorning mos koordinatalarini qo'ysak $z=f(x,y)$ sirtga $M_0(x_0,y_0,z_0)$ nuqtada o'tkazilgan normal tenglamasi

$$\frac{x-x_0}{f'_x(x_0,y_0)}=\frac{y-y_0}{f'_y(x_0,y_0)}=\frac{z-z_0}{-1} \qquad (51.7)$$

hosil bo'ladi.

Agarda sirt $F(x,y,z)=0$ tenglama bilan berilgan bo'lsa, u holda (51.6) dan urinma tekislikning normal vektori

$$\vec{n}=\{F'_x(M_0),F'_y(M_0),F'_z(M_0)\}$$

bo'lishi kelib chiqadi. Ana shu vektorning o'zi sirtga $M_0(x_0,y_0,z_0)$ nuqtada normalning yo'naltiruvchi vektori bo'ladi. Shuning uchun normalning bu holdagi tenglamasi

$$\frac{x-x_0}{F'_x(M_0)}=\frac{y-y_0}{F'_y(M_0)}=\frac{z-z_0}{F'_z(M_0)} \qquad (51.8)$$

bo'ladi.

6-misol. $z=4x^2+6y^2$ sirtga $M_0(2;1;22)$ nuqtada normal tenglamasi yozilsin.

Yechish. $f(x,y)=4x^2+6y^2$ deb belgilab shu funksiyaning xususiy hosilalarining $P_0(2;1)$ nuqtadagi qiymatlarini topamiz: $x_0=2$, $y_0=1$, $z_0=22$

$$f'_x(x,y)=8x,\quad f'_y(x,y)=12y,\quad f'_x(P_0)=f'_x(2;1)=8\cdot 2=16,$$
$$f'_y(P_0)=f'_x(2;1)=12\cdot 1=12.$$

Bularni (51.7) ga qo‘yib normal tenglamasini hosil qilamiz:

$$\frac{x-2}{16}=\frac{y-1}{12}=\frac{z-22}{-1}.$$

7-misol. $x^2+y^2+z^2=49$ sferaga $M_0(2;3;6)$ nuqtada o‘tkazilgan normal tenglamasi topilsin.

Yechish. $F(x,y,z)=x^2+y^2+z^2-49$ deb belgilasak

$$F'_x=2x,\quad F'_y=2y,\quad F'_z=2z,\quad F'_x(M_0)=2\cdot 2=4,$$
$$F'_y(M_0)=2\cdot 3=6,\quad F'_z(M_0)=2\cdot 6=12.$$

bo‘ladi. Bularni, hamda $x_0=2$, $y_0=3$, $z_0=6$ qiymatlarni (51.8) formulaga qo‘ysak normal tenglamasi

$$\frac{x-2}{4}=\frac{y-3}{6}=\frac{z-6}{12}$$

yoki

$$\frac{x-2}{2}=\frac{y-3}{3}=\frac{z-6}{6}$$

hosil bo‘ladi.

Endi ikki o‘zgaruvchi funksiyasining to‘la differensialining geometrik ma‘nosi bilan tanishamiz. Bir o‘zgaruvchi funksiyasining differensiali egri chiziqqa o‘tkazilgan urinmaning urinish nuqtasi ordinatasining orttirmasini anglatishi aytilgan edi. Shunga o‘xshash ikki o‘zgaruvchi funksiyasining to‘la differensiali sirtga o‘tkazilgan urinma tekislikning urinish nuqtasi applikasining orttirmasini ifodalaydi. Bu to‘la differensialning geometrik ma‘nosi.

O'z-o'zini tekshirish uchun savollar

1. Bir o'zgaruvchining oshkormas funksiyasini ta'riflang.
2. Ikki o'zgaruvchining oshkormas funksiyasini ta'riflang.
3. Bir o'zgaruvchining oshkormas funksiyasinining mavjudligi haqidagi teoremani ayting.
4. Ikki o'zgaruvchining oshkormas funksiyasining mavjudligi haqidagi teoremani ayting.
5. Sirtga urinma tekislikni ta'riflang.
6. Sirtga normalni ta'riflang.
7. $z=f(x,y)$ sirtga urinma tekislik tenglamasini yozing.
8. $F(x,y,z)=0$ sirtga urinma tekislik tenglamasini yozing.
9. $z=f(x,y)$ sirtga normal tenglamasini yozing.
10. $F(x,y,z)=0$ sirtga normal tenglamasini yozing.

Mustaqil yechish uchun mashqlar

Quyidagi funksiyalarning hosilalari topilsin.

1. $x^2+y^2=4$. Javob: $y'=\frac{-x}{y}$.

2. $e^y-e^x+xy=0$. Javob: $y'=\frac{e^x-y}{e^x+x}$.

3. $\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0$. Javob: $y'=-\left(\frac{b}{a}\right)^2\frac{x}{y}$.

4. $y^x=x^y$. Javob: $\frac{yx^{y-1}-y^x\ell ny}{xy^{x-1}-x^y\ell nx}$.

Quyidagi tenglamalar bilan berilgan x va y ning oshkormas funksiyasi z ning xususiy hosilalari topilsin.

5. $x^2+y^2+z^2-R^2=0$. Javob: $z'_x=-\frac{x}{z}$, $z'_y=-\frac{y}{z}$.

6. $x-ytgz=0$. Javob: $z'_x=-\frac{\cos^2 z}{y}$, $z'_y=-\frac{\sin 2z}{2y}$.

7. $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ giperboloidga (x_0, y_0, z_0) nuqtada urinma tekislik tenglamasini yozing. Javob: $\frac{x_0x}{a^2}-\frac{y_0y}{b^2}-\frac{z_0z}{c^2}=1$.

8. $x^2+2y^2+z^2=1$ sirtga (ellipsoidga) $x-y+2z=0$ tekislikka parallel bo'lgan urinma tekislik o'tkazilsin. Javob: $x-y+2z=\pm\sqrt{\frac{11}{2}}$.

9. $x^2-4y^2+2z^2=6$ sirtga $(2;2;3)$ nuqtada normal tenglamasi topilsin. Javob: $\frac{x-0}{1}=\frac{y-10}{-4}=\frac{z+3}{3}$.

10. $z=2x^2+4y^2$ sirtga $M=(2;1;12)$ nuqtada normal tenglamasi topilsin. Javob: $\frac{x-2}{8}=\frac{y-1}{8}=\frac{z-12}{-1}$.

52. YUQORI TARTIBLI XUSUSIY HOSILALAR VA DIFFERENSIALLAR

52.1. Yuqori tartibli xususiy hosilalar

$z=f(x,y)$ funksiya $P\ (x,y)$ nuqtada va uning biror atrofida aniqlangan bo'lsin. Bu atrofning har bir nuqtasida

$$\frac{\partial z}{\partial x}=f'_x(x,y),\quad \frac{\partial z}{\partial y}=f'_y(x,y)$$

xususiy hosilalar mavjud bo'lsin. Ularni **birinchi tartibli** (yoki **birinchi**) **xususiy hosilalar** deb ataymiz. Bu xususiy hosilalar ham x va y erkli o'zgaruvchilarning funksiyalari bo'lishi ravshan.

1-ta'rif. $z=f(x,y)$ funksiyaning birinchi tartibli xususiy hosilalaridan x va y o'zgaruvchilar bo'yicha olingan xususiy hosilalar, agar ular mavjud bo'lsa, $z=f(x,y)$ funksiyaning $P\ (x,y)$ nuqtadagi **ikkinchi tartibli** (yoki ikkinchi) **xususiy hosilalari** deb ataladi va quyidagicha belgilanadi:

$\frac{\partial^2 z}{\partial x^2}=f''_{xx}(x,y)=f''_{x^2}(x,y),$ bu yerda $f(x,y)$ funksiya ketma-ket ikki marta x bo'yicha differensillanadi;

$\frac{\partial^2 z}{\partial x\partial y}=f''_{xy}(x,y),$ bu yerda $f(x,y)$ funksiya avval x bo'yicha va so'ngra natija y bo'yicha differensiallanadi;

$\frac{\partial^2 z}{\partial y\partial x}=f''_{yx}(x,y),$ bu yerda $f(x,y)$ funksiya avval y bo'yicha va so'ngra natija x bo'yicha differensiallanadi;

$\frac{\partial^2 z}{\partial y^2}=f''_{yy}(x,y)=f''_{y^2}(x,y),$ bu yerda $f(x,y)$ funksiya ketma-ket ikki marta y bo'yicha differensillanadi;

$\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}$ xususiy hosilaning har biri x va y o'zgaruvchilarning funksiyalari bo'lganligi uchun ularni x va y bo'yicha differensiallash mumkin. Shuning uchun ikki o'zgaruvchi funksiyasining ikkinchi tartibli xususiy hosilalari to'rtta bo'ladi.

$f''_{xy}(x,y)$ va $f''_{yx}(x,y)$ ikkinchi tartibli xususiy hosilalar **aralash xususiy hosilalar** deyiladi.

1-misol. $z=x^3+5x^2y^3+6xy+4$ funksiyaning barcha ikkinchi tartibli xususiy hosilalari topilsin.

Yechish.

$$\frac{\partial z}{\partial x}=z'_x=(x^3+5x^2y^3+6xy+4)'_x=3x^2+10xy^3+6y,$$

$$\frac{\partial z}{\partial y}=z'_y=(x^3+5x^2y^3+6xy+4)'_y=15x^2y^2+6x,$$

$$\frac{\partial^2 z}{\partial x^2}=z''_{x^2}=(3x^2+10xy^3+6y)'_x=6x+10y^3,$$

$$\frac{\partial^2 z}{\partial x\partial y}=z''_{xy}=(3x^2+10xy^3+6y)'_y=30xy^2+6,$$

$$\frac{\partial^2 z}{\partial y\partial x}=z''_{yx}=(15x^2y^2+6x)'_x=30xy^2+6,$$

$$\frac{\partial^2 z}{\partial y^2}=z''_{y^2}=(15x^2y^2+6x)'_y=30x^2y.$$

2-misol. $z=\cos x\cdot\sin y$ funksiyaning aralash xususiy hosilalari topilsin.

Yechish.

$$z'_x=-\sin x\cdot\sin y,\quad z'_y=\cos x\cdot\cos y,$$

$$z''_{xy}=-\sin x\cdot\cos y,\quad z''_{yx}=-\sin x\cdot\cos y.$$

Har ikkala misollarda ham $f''_{xy}(x,y)$, $f''_{yx}(x,y)$ aralash hosilalar o'zaro teng, ya'ni ular differensiallash tartibiga bog'liq emas.

2-ta'rif. $z=f(x,y)$ funksiyaning ikkinchi tartibli xususiy hosilalaridan x va y o'zgaruvchilar bo'yicha olingan xususiy hosilalar, agar ular mavjud bo'lsa, $z=f(x,y)$ funksiyaning $P(x,y)$ nuqtadagi **uchinchi**

tartibli (yoki uchunchi) **xususiy hosilalari** deb ataladi. Ular sakkizta bo‘ladi:

$\frac{\partial^3 z}{\partial x^3} = f'''_{x^3}(x,y)$, bu yerda *f(x,y)* funksiya ketma-ket uch marta *x* bo‘yicha differentsillanadi;

$\frac{\partial^3 z}{\partial x^2 \partial y} = f'''_{x^2 y}(x,y)$, bu yerda *f(x,y)* funksiya avval ketma-ket ikki marta *x* bo‘yicha va so‘ngra natija *y* bo‘yicha differensiallanadi;

$\frac{\partial^3 z}{\partial x \partial y \partial x} = f'''_{xyx}(x,y)$, bu yerda *f(x,y)* funksiya avval *x* bo‘yicha, so‘ngra $\frac{\partial z}{\partial x}$ xususiy hosila *y* bo‘yicha va natija yana *x* bo‘yicha differensiallanadi;

$\frac{\partial^3 z}{\partial x \partial y^2} = f'''_{xy^2}(x,y)$, bu yerda *f(x,y)* funksiya avval *x* bo‘yicha, so‘ngra $\frac{\partial z}{\partial x}$ ikki marta ketma-ket *y* bo‘yicha differentsillanadi;

Shunga o‘xshash

$$\frac{\partial^3 z}{\partial y \partial x^2} = f'''_{yx^2}(x,y), \quad \frac{\partial^3 z}{\partial y \partial x \partial y} = f'''_{yxy}(x,y),$$

$$\frac{\partial^3 z}{\partial y^2 \partial x} = f'''_{y^2 x}(x,y), \quad \frac{\partial^3 z}{\partial y^3} = f'''_{y^3}(x,y)$$

xususiy hosilalar ham *z=f(x,y)* funksiyani ma‘lum tartibda *x* va *y* o‘zgaruvchilar bo‘yicha differensiallash natijasida hosil bo‘ladi.

3-misol. $z=x^5+5xy^3-3xy^2+y^6$ funksiyaning uchunchi tartibli xususiy hosilalari topilsin.

Yechish. $f(x,y)=x^5+5xy^3-3xy^2+y^6$ belgilashni kiritamiz.

U holda:

$$f_x'(x,y)=5x^4+5y^3-3y^2,\quad f_y'(x,y)=15xy^2-6xy+6y^5,$$

$$f''_{x^2}(x,y)=20x^3,\quad f''_{xy}(x,y)=15y^2-6y,\quad f''_{yx}(x,y)=15y^2-6y,$$

$$f''_{y^2}(x,y)=30xy-6x+30y^4,\quad f'''_{x^3}(x,y)=60x^2,\quad f'''_{x^2y}(x,y)=0,$$

$$f'''_{xyx}(x,y)=0,\quad f'''_{xy^2}(x,y)=30y-6,\quad f'''_{yx^2}(x,y)=0,$$

$$f'''_{yxy}(x,y)=30y-6,\quad f'''_{y^2x}(x,y)=30y-6,\quad f'''_{y^3}(x,y)=30x+120y^3.$$

Bunda

$$f'''_{x^2y}(x,y)=f'''_{xyx}(x,y)=f'''_{yx^2}(x,y)=0$$

va

$$f'''_{xy^2}(x,y)=f'''_{yxy}(x,y)=f'''_{y^2x}(x,y)=30y-6.$$

Bu misolda ham aralash xususiy hosilalar o'zaro teng, ya'ni xususiy hosilalarning qiymati funksiyani differensiallash tartibiga bog'liq emas.

Umuman olganda aralash xususiy hosilalar funksiyani differensiallash tartibiga bog'liq bo'ladi. Masalan

$$f(x,y)=\begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2} & \text{agar} \quad x^2+y^2\neq 0 \quad \text{bo' lsa,} \\ 0 \quad \text{agar} \quad x^2+y^2=0 \quad \text{bo' lsa} \end{cases}$$

funksiya $(0,0)$ nuqtada o'zaro teng bo'lmagan $f''_{xy}(0;0), f''_{yx}(0;0)$ xususiy hosilalarga ega.

Haqiqatan

$$f_x'(x,y)=\begin{cases} y\dfrac{y(x^4-y^4+4x^2y^2)}{(x^2+y^2)^2} & \text{agar} \quad x^2+y^2\neq 0 \quad \text{bo' lsa,} \\ 0 \quad \text{agar} \quad x^2+y^2=0 \quad \text{bo' isa.} \end{cases}$$

Shuning uchun

$$f''_{xy}(0;0)=\lim_{\Delta y\to 0}\frac{f_x'(0;0+\Delta y)-f_x'(0;0)}{\Delta y}=\lim_{\Delta y\to 0}\frac{-y^4}{(y^2)^2}=-1.$$

Shunga o'xshash hisoblashlarni bajarib $f''_{yx}(0;0)=1$ ekanini aniqlash mumkin. Demak

$$f''_{xy}(0;0) \neq f''_{yx}(0;0).$$

Shunday qilib aralash xususiy hosilalar har doim ham o'zaro teng bo'lavermas ekan.

Qanday shartlar bajarilganda aralash xususiy hosilalarning qiymati funksiyani differensiallash tartibiga bog'liq bo'lmaydi degan savolga quyidagi teorema javob beradi. Biz uni isbotsiz keltiramiz.

52.1-teorema. Agar $z=f(x,y)$ funksiyaning barcha aralash xususiy hosilalari $P(x,y)$ nuqtaning biror δ-atrofida mavjud va shu nuqtaning o'zida uzluksiz bo'lsa, u holda aralash xususiy hosilalarning qiymati funksiyani differensiyaallash tartibiga bog'liq bo'lmaydi, ya'ni

$$f''_{xy}(x,y)=f''_{yx}(x,y),\quad f'''_{x^2y}(x,y)=f'''_{xyx}(x,y)=f'''_{yx^2}(x,y),$$
$$f'''_{xy^2}(x,y)=f'''_{yxy}(x,y)=f'''_{y^2x}(x,y).$$

3-ta'rif. $z=f(x,y)$ funksiyaning $(n-1)$-tartibli xususiy hosilalaridan x va y o'zgaruvchilar bo'yicha olingan xususiy hosilalar, agar ular mavjud bo'lsa, $z=f(x,y)$ funksiyaning $P(x,y)$ nuqtadagi ***n*- tartibli** (yoki n-) **xususiy hosilalari** deb ataladi. Ular 2^n ta bo'ladi. Masalan,

$$\frac{\partial^n z}{\partial x^p \partial y^{n-p}}=z^{(n)}_{x^p y^{n-p}}$$ ifoda n-tartibli xususiy hosila, bu yerda z funksiya avval x bo'yicha kema-ket p marta differensiallangan so'ngra natija kema-ket y bo'yicha $n-p$ marta differensillangan. Istalgan sondagi o'zgaruvchilar funksiyasining yuqori tartibli xususiy hosilalari ham shunga o'xshash ta'riflanadi.

52.1-teorema n-tartibli aralash xususiy hosilalar uchun ham o'rinlidir.

52.2. Yuqori tartibli to'la differensiallar

49-mavzuda $P(x,y)$ nuqtada differensillanuvchi $z=f(x,y)$ funksiyaning to'la differensiali tushunchasi kiritilgan va u

$$dz = f'_x(x,y)dx + f'_y(x,y)dy \qquad (52.1)$$

formula yordamida topilishi ko'rsatilgan edi.

dz ni birinchi tartibli(yoki birinchi) to'la differensial deb ataymiz.

Faraz qilaylik $f'_x(x,y)$, $f'_y(x,y)$ funksiyalar $P(x,y)$ nuqtada differensiallanuvchi bo'lsin. U holda dx, dy differensiallarni o'zgarmasligi hisobga olinsa dz to'la differensial ikki o'zgaruvchi x va y ning $P(x,y)$ nuqtada differensiallanuvchi funksiyasidan iborat bo'ladi va uning to'la differensiali (52.1) formulaga binoan

$$d(dz) = d(f'_x(x,y)dx + f'_y(x,y)dy) = (f'_x(x,y)dx + f'_y(x,y)dy)'_x dx + \\ + (f'_x(x,y)dx + f'_y(x,y)dy)'_y dy \qquad (52.2)$$

kabi topiladi.

4-ta'rif. $z=f(x,y)$ funksiyaning $P(x,y)$ nuqtadagi to'la differensiali dz ning to'la differensiali (agar u mavjud bo'lsa) shu funksiyaning $P(x,y)$ nuqtadagi **ikkinchi tartibli** (yoki **ikkinchi**) **to'la differensiali** deb ataladi va d^2z kabi belgilanadi.

5-ta'rif. $z=f(x,y)$ funksiyaning $P(x,y)$ nuqtadagi ikkinchi tartibli to'la differensiali d^2z ning to'la dfferentsiali (agar u mavjud bo'lsa) shu funksiyaning $P(x,y)$ nuqtadagi **uchinchi tartibli** (yoki uchinchi) **to'la differensiali** deb ataladi va d^3z kabi belgilanadi.

$z=f(x,y)$ funksiyaning n-tartibli (yoki n-) to'la differensiali ham shunga o'xshash ta'riflanadi va d^nz kabi belgilanadi.

To'la differensial hamda n-tartibli to'la differensialning ta'rifidan $P(x,y)$ nuqtada n-tartibli to'la differensialga ega $z=f(x,y)$ funksiya shu nuqtada n-tartibligacha uzluksiz barcha xususiy hosilalarga ega ekanligi kelib chiqadi.

Endi yuqori tartibli to'la differensiallarni hisoblash uchun formulalar chiqaramiz.

(52.2) formuladan:

$$d^2z=f''_{x^2}(x,y)(dx)^2+f''_{yx}(x,y)dydx+f''_{xy}(x,y)dxdy+f''_{y^2}(x,y)(dy)^2.$$

Shartga ko'ra $f''_{xy}(x,y)$ va $f''_{yx}(x,y)$ aralash xususiy hosilalar $P(x,y)$ nuqtada uzluksiz bo'lganliklari uchun 52.1-teoremaga binoan ular o'zaro teng.

Demak,

$$d^2z=f''_{x^2}(x,y)(dx)^2+2f''_{yx}(x,y)dxdy+f''_{y^2}(x,y)(dy)^2. \tag{52.3}$$

Agar

$$f''_{x^2}(x,y)(dx)^2+2f''_{xy}(x,y)dxdy+f''_{y^2}(x,y)(dy)^2=\left(\frac{\partial}{\partial x}dx+\frac{\partial}{\partial y}dy\right)^2 f(x,y)$$

belgilashni kiritsak ikkinchi tartibli to'la differensialni hisoblash uchun

$$d^2z=\left(\frac{\partial}{\partial x}dx+\frac{\partial}{\partial y}dy\right)^2 f(x,y) \tag{52.3r}$$

formulani hosil qilamiz.

Shunga o'xshash

$$d^nz=\left(\frac{\partial}{\partial x}dx+\frac{\partial}{\partial y}dy\right)^n f(x,y) \tag{52.4}$$

formulaning to'g'riligini ko'rsatish mumkin.

Tenglikning o'ng tomonidagi ifoda ikki hadning n-darajasini Nyuton binom formulasi bo'yicha yoyilmasini eslatadi:

$$d^nz=f^{(n)}_{x^n}(x,y)(dx)^n+nf^{(n)}_{x^{n-1}y}(x,y)(dx)^{n-1}dy+$$

$$+\frac{n(n-1)}{2}f^{(n)}_{x^{n-2}y^2}(x,y)(dx)^{n-2}(dy)^2+\ldots+f^{(n)}_{y^n}(x,y)(dy)^n.$$

4-misol. $z=\operatorname{arctg}\dfrac{y}{x}$ funksiyaning ikkinchi tartibli to'la differensiali topilsin.

Yechish.

$$f(x,y)=\operatorname{arctg}\frac{y}{x},\quad f'_x(x,y)=\frac{\left(\frac{y}{x}\right)'_x}{1+\left(\frac{y}{x}\right)^2}=\frac{-\frac{y}{x^2}}{1+\frac{y^2}{x^2}}=-\frac{y}{x^2+y^2},$$

$$f'_y(x,y)=\frac{\left(\frac{y}{x}\right)'_y}{1+\left(\frac{y}{x}\right)^2}=\frac{\frac{1}{x}}{1+\frac{y^2}{x^2}}=-\frac{x}{x^2+y^2},$$

$$f''_{x^2}(x,y)=-\frac{-y(x^2+y^2)'_x}{(x^2+y^2)^2}=\frac{2xy}{(x^2+y^2)^2},$$

$$f''_{y^2}(x,y)=-\frac{2xy}{(x^2+y^2)^2},$$

$$f''_{xy}(x,y)=\left(-\frac{y}{x^2+y^2}\right)'_y=-\frac{x^2+y^2-y\cdot 2y}{(x^2+y^2)^2}=\frac{y^2-x^2}{(x^2+y^2)^2}.$$

Topilgan qiymatlarni (52.3) formulaga qo'ysak

$$d^2z=\frac{2xy(dx)^2+2(y^2-x^2)dxdy-2xy(dy)^2}{(x^2+y^2)^2}$$

kelib chiqadi.

5-misol. $z=\sin x\cos y$ funksiyaning uchinchi tartibli to'la differensiali topilsin.

Yechish. (52.4) formula $n=3$ bo'lganda

$$d^3z=f'''_{x^3}(x,y)(dx)^3+3f'''_{x^2y}(x,y)(dx)^2dy+$$
$$+3f'''_{xy^2}(x,y)dx(dy)^2+f'''_{y^3}(x,y)(dy)^3 \qquad (52.5)$$

ko'rinishga ega bo'ladi.

$$f(x,y)=\sin x\cos y,\quad f'_x(x,y)=\cos x\cos y,\quad f'_y(x,y)=-\sin\sin y,$$

$$f''_{x^2}(x,y)=-\sin x\cos y,\quad f''_{y^2}(x,y)=-\sin x\cos y,\quad f''_{xy}(x,y)=-\cos x\sin y,$$

$$f'''_{x^3}(x,y)=-\cos x\cos y,\quad f'''_{y^3}(x,y)=\sin x\sin y,$$

$$f'''_{x^2y}(x,y)=\sin x\sin y, f'''_{xy^2}(x,y)=-\cos x\cos y.$$

Topilgan qiymatlarni (52.5) formulaga qo'ysak

$$d^3z=-\cos x\cos y(dx)^3+3\sin x\sin y(dx)^2dy-$$

$$-3\cos x\cos ydx(dy)^2+\sin x\sin y(dy)^3$$

hosil bo'ladi.

O'z-o'zini tekshirish uchun savollar

1. Ikki o'zgaruvchi funksiyasining birinchi tartibli xususiy hosilalarini ta'riflang.
2. Ikki o'zgaruvchi funksiyasining ikkinchi tartibli xususiy hosilalarini ta'riflang.
3. Ikki o'zgaruvchi funksiyasining n-tartibli xususiy hosilasini ta'riflang. Ular nechta?
4. Aralash hosilalarning tengligi haqidagi teoremani ayting.
5. Aralash hosilalari o'zaro teng bo'lmagan funksiyaga misol keltiring.
6. Istalgan sondagi o'zgaruvchilarning funksiyasini yuqori tartibli xususiy hosilalari qanday ta'riflanadi?
7. Ikki o'zgaruvchi funksiyasining birinchi tartibli to'la differensialini ta'riflang.
8. Ikki o'zgaruvchining funksiyasi uchun ikkinchi tartibli to'la differensialni ta'riflang va uni topish formulasini yozing.
9. Ikki o'zgaruvchining funksiyasi uchun n-tartibli to'la differensialni ta'riflang va uni hisoblash formulasini yozing.

Mustaqil yechish uchun mashqlar

Quyidagi funksiyalarning ikkinchi tartibli xususiy hosilalari hisoblansin.:

1. $z=x^3-4x^2y+5y^2$.

Javob: $\frac{\partial^2 z}{\partial x^2}6x-8y$; $\frac{\partial^2 z}{\partial x\partial y}=-8x$; $\frac{\partial^2 z}{\partial y^2}=10$.

2. $z=e^x\ln y+\sin y\ln x$.

Javob: $\frac{\partial^2 z}{\partial x^2}=e^x\ln y-\frac{\sin y}{x^3}$, $\frac{\partial^2 z}{\partial x\partial y}=\frac{e^x}{y}+\frac{\cos y}{x}$,

$\frac{\partial^2 z}{\partial y^2}=-\frac{e^x}{y^2}-\sin y\cdot\ln x$.

3. Agar $u=\frac{1}{\sqrt{x^2+y^2+z^2}}$ bo'lsa, $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0$ bo'lishi isbotlansin.

4. Agar $z=\frac{x^2y^2}{x+y}$ bo'lsa, $x\frac{\partial^2 z}{\partial x^2}+y\frac{\partial^2 z}{\partial x\partial y}=2\frac{\partial z}{\partial x}$ bo'lishi isbotlansin.

5. Agar $z=\ln(x^2+y^2)$ bo'lsa, $\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0$ bo'lishi isbotlansin.

6. $z=\operatorname{arctg}\frac{x}{y}$ funksiya uchun d^2z topilsin.

Javob: $d^2z=\frac{-2xy(dx)^2+2(x^2-y^2)dxdy+2xy(dy)^2}{(x^2+y^2)^2}$.

7. $z=\sin(2x+y)$ funksiya uchun d^2z topilsin.

Javob: $d^3z=-\cos(2x+y)[8(dx)^3+12(dx)^2dy+6dx(dy)^2+(dy)^3]$.

Haqiqatan, $f(0,0)=1$. $(0;0)$ nuqtaning $x^2+y^2=\frac{\pi}{2}$ aylana bilan chegaralangan atrofini qaraymiz. Bu atrofning istalgan $(0;0)$ dan farqli $P(x,y)$ nuqtasi uchun $\sin(x^2+y^2)>0$ bo'ladi, chunki $0<x^2+y^2<\frac{\pi}{2}$. Shuning uchun

$$f(x,y)=1-\sin(x^2+y^2)<1, \text{ ya'ni } f(x,y)<f(0;0).$$

53.2. Ekstremum mavjudligining zaruriy shartlari

53.1-teorema (ekstremum mavjudligining zaruriy shartlari)

Agar $z=f(x,y)$ funksiya $P_o(x_o,y_o)$ ekstremum nuqtasida differensiallanuvchi bo'lsa, u holda uning shu nuqtadagi xususiy hosilalari nolga teng, ya'ni

$$f'_x(x_o,y_o)=0, \quad f'_y(x_o,y_o)=0$$

bo'ladi.

Isboti. $z=f(x,y)$ funksiyaning P_o nuqtadagi x bo'yicha xususiy hosilasi $f'_x(x_o,y_o)$ bir o'zgaruvchi funksiyasi $g(x)=f(x,y_0)$ ning $x=x_o$ dagi hosilasidan iborat. Ammo $x=x_o$ da $g(x)$ ekstremumga ega bo'lganligi sababli bir o'zgaruvchi funksiyasi ekstrimumi mavjudligining zaruriy shartiga binoan $g'(x_o)=0$, $f'_x(x_o,y_o)=0$ bo'ladi.

$f'_y(x_o,y_o)=0$ tenglik ham shunga o'xshash isbotlanadi. Shunday qilib differensiyallanuvchi funksiya birinchi tartibli xususiy hosilalari nolga teng bo'lgan nuqtalaridagina ekstremumga ega bo'lishi mumkin ekan.

Shuni ta'kidlash lozimki funksiya ekstremumga birinchi tartibli xususiy hosilalari nolga teng bo'lgan nuqtalardan tashqari ular mavjud bo'lmagan nuqtalarda ham ega bo'lishi mumkin. Masalan, $z=\sqrt{x^2+y^2}$ funksiyaning xususiy hosilalari

$$z'_x=\frac{x}{\sqrt{x^2+y^2}}, \quad z'_y=\frac{y}{\sqrt{x^2+y^2}}$$

$0(o;o)$ nuqtada mavjud emas, ammo funksiya bu nuqtada minimumga ega.

3-ta'rif. $z=f(x,y)$ funksiyaning birinchi tartibli xususiy hosilalari $f'_x(x,y)$, $f'_y(x,y)$ lar nolga aylanadigan va mavjud bo'lmaydigan nuqtalar funksiyaning **kritik** nuqtalari deyiladi.

Funksiya ekstrimumga faqatgina o'zining kritik nuqtalaridagina erishishi mumkinligini ta'kidlash o'rinli.

Funksiyaning har qanday kritik nuqtasi ham uning ekstrimum nuqtasi bo'lavermaydi. Masalan, $z=f(x,y)=xy$ funksiyaning birinchi tartibli xususiy hosilalari $f'_x(x,y)=y$, $f'_y(x,y)=x$ $P_o(0,0)$ nuqtada nolga teng, ya'ni bu nuqta funksiyaning kritik nuqtasi. Biroq funksiya bu kritik nuqtada ekstremumga ega emas. Haqiqatan, $f(P_o)=0$ ammo $P_o(0,0)$ nuqtaning istalgan atrofida $f(x,y)=xy$ funksiya ham musbat (I,III chorak nuqtalari uchun), ham manfiy (II, IV chorak nuqtalari uchun) qiymatlarni qabul qiladi.

53.3. Ekstremum mavjudligining yetarlilik shartlari

Ikki o'zgaruvchi funksiyasini kritik nuqtasida ekstremum mavjudligini tekshirish imkonini beruvchi teoremani isbotsiz keltiramiz.

53.2-teorema. $z=f(x,y)$ funksiyaning $P_o(x_o,y_o)$ kritik nuqtasi uning ekstremum nuqtasi bo'lishi uchun

$$\Delta(P_o)=f''_{x^2}(P_o)\cdot f''_{y^2}(P_o)-[f''_{xy}(P_o)]^2>0$$

shartning bajarilishi yetarlidir. Bunda: $f''_{x^2}(P_o)<0$ bo'lganda $P_o(x_o,y_o)$ funksiyaning maksimum nuqtasi, $f''_{x^2}(P_0)>0$ bo'lganda $P_0(x_0,y_0)$ funksiyaning minimum nuqtasi bo'ladi. $\Delta(P_o)<0$ shart $P_o(x_o,y_o)$ kritik nuqta funksiyaning ekstremum nuqtasi bo'lmasligi uchun yetarli, ya'ni bu holda $P_o(x_o,y_o)$ nuqtada funksiya ekstremumga ega bo'lmaydi.

$\Delta(P_o)=0$ bo'lganda masala ochiq qoladi, ya'ni bu holda $z=f(x,y)$ funksiya $P_o(x_o,y_o)$ kritik nuqtada ekstremumga ega bo'lishi ham bo'lmasligi ham mumkin. Barcha ikkinchi tartibli xususiy hosilalar $P_o(x_o,y_o)$ nuqtada uzluksiz deb faraz qilinadi.

Ikki o'zgaruvchi funksiyasi $z=f(x,y)$ ning ekstremumini topish quyidagi sxema asosida amalga oshiriladi.

1. Funksiyaning barcha kritik nuqtalari topiladi.

Buning uchun:

a) funksiyaning birinchi tartibli xususiy hosilalari topiladi.

b) Topilgan xususiy hosilalar nolga tenglashtirilib hosil bo'lgan

$$\left.\begin{array}{l} f'_x(x,y)=0, \\ f'_y(x,y)=0 \end{array}\right\}$$

sistema yechiladi. Koordinatalari shu sistemaning yechimidan iborat nuqtalar funksiyaning kritik nuqtalari bo'ladi.

c) Funksiyaning birinchi tartibli xususiy hosilalari mavjud bo'lmagan nuqtalar topiladi.

2. Topilgan har bir kritik nuqtada 53.2-teoremadan foydalanib funksiya tekshiriladi.

3-misol. $f(x,y)=x^3+3xy^2-30x-18y$ funksiyaning ekstremumlarini toping.

Yechish. Birinchi tartibli xususiy hosilalarni topamiz:

$f'_x(x,y)=(x^3)'_x+3y^2\cdot x'_x-30x'_x-(18y)'_x=3x^2+3y^2\cdot 1-30\cdot 1-0=3(x^2+y^2-10),$

$f'_y(x,y)=(x^3)'_y+3x(y^2)'_y-(30x)'_y-(18\cdot y)'_y=0+6xy-0-18=3(2xy-6).$

Ushbu hosilalarni nolga tenglashtirib

$$\begin{cases} 3(x^2+y^2-10)=0, \\ 3(2xy-6)=0 \end{cases} \quad \text{yoki} \quad \begin{cases} x^2+y^2=10, \\ 2xy=6. \end{cases}$$

sitemaga ega bo'lamiz. Oxirgi sistemaning tenglamalarini hadlab qo'shsak va hadlab ayirsak

$$\begin{cases} x^2+2xy+y^2=16, \\ x^2-2xy+y^2=4 \end{cases} \quad \text{yoki} \quad \begin{cases} (x+y)^2=16, \\ (x-y)^2=4. \end{cases}$$

sistema hosil bo'ladi. Bundan

$$\begin{cases} x+y=\pm 4, \\ x-y=\pm 2 \end{cases}$$

sistemani hosil qilamiz. Sistemaning tenglamalarini hadlab qo'shsak $2x=\pm 4\pm 2$ yoki $x=\pm 2\pm 1$, birinchi tenglamadan ikkinchisini hadlab ayirsak $2y=\pm 4\mp 2$ yoki $y=\pm 2\mp 1$ bo'ladi. Demak, $x_1=3$, $x_2=2-1=1$, $x_3=-2+1=-1$, $x_4=-2-1=-3$, $y_1=2-1=1$, $y_2=2+1=3$, $y_3=-2-1=-3$, $y_4=-2+1=-1$ va $P_1(3;1)$, $P_2(1;3)$ $P_3(-1;-3)$ $P_4(-3;-1)$ kritik nuqtalar.

Endi ikkinchi tartibli xususiy hosilalarni topamiz:

$$f''_{x^2}(x,y)=(f'_x(x,y))'_x=(3(x^2+y^2-10))'_x=3\cdot 2x=6x,$$

$$f''_{xy}(x,y)=(f'_x(x,y))'_y=(3(x^2+y^2-10))'_y=6y,$$

$$f''_{y^2}(x,y)=(f'_y(x,y))'_y=(6xy-18)'_y=6x,$$

$$\Delta(P)=f''_{x^2}(P)\cdot f''_{y^2}(P)-[f''_{xy}(P)]^2=6x\cdot 6x-(6y)^2=36(x^2-y^2)$$

ifodani tuzamiz.

1) $\Delta(P_1)=36(3^2-1^2)=288>0$, $f''_{x^2}(P_1)=6\cdot 3=18>0$

bo'lgani uchun $P_1(3;1)$ kritik nuqta funksiyaning minimum nuqtasi.

2) $\Delta(P_2)=36(1^2-3^2)<0$ bo'lgani uchun $P_2(1;3)$ kritik nuqtada ekstremum yo'q.

3) $\Delta(P_3)=36((-1)^2-(-3)^2)36\cdot(-8)<0$ bo'lgani uchun $P_3(-1;-3)$ kritik nuqtada ham ekstremum yo'q.

4) $\Delta(P_4)$ $36((-3)^2-(-1)^2)=288>0$, $f''_{x^2}(P_4)=6\cdot(-1)=-6<0$ bo'lgani uchun $P_4(-3;-1)$ kritik nuqtada funksiya maksimumga ega.

Shunday qilib berilgan funksiya $P_1(3;1)$ kritik nuqtada $f(P_1)=f(3;1)=3^3+3\cdot 3\cdot 1^2-30\cdot 3-18\cdot 1=-72$ minimumga va $P_4(-3;-1)$ nuqtada $f(P_4)=f(-3;-1)=(-3)^3+3(-3)\cdot(-1)^2-30\cdot(-3)-18\cdot(-1)=72$ maksimumga ega.

53.4. Ikki o'zgaruvchi funksiyasining yopiq sohadagi eng katta va eng kichik qiymatlari

Faraz qilaylik $z=f(x,y)$ funksiya yopiq chegaralangan G sohada uzluksiz va sohaning ichida differensiallanuvchi bo'lsin.

Yopiq chegaralangan sohada uzluksiz funksiya shu sohada o'zining eng katta va eng kichik qiymatlariga erishishi ta'kidlangan edi. Agar funksiya eng katta va eng kichik qiymatlarini *G* sohaning ichki nuqtasida qabul qilsa, u holda bu nuqta $z=f(x,y)$ funksiyaning ekstremum nuqtasi bo'lishi ravshan. Bundan tashqari funksiya eng katta va eng kichik qiymatlariga G sohaning chegara nuqtalarida erishishi ham mumkin.

Shuning uchun yopiq chegaralangan *G* sohada berilgan $z=f(x,y)$ funksiyaning *G* sohadagi eng katta va eng kichik qiymatlari quyidagicha topiladi. Funksiyaning *G* sohadagi barcha kritik nuqtalari hamda funksiyaning bu nuqtalardagi qiymatlari topiladi. Keyin funksiyaning *G* sohaning chegarasidagi eng katta va eng kichik qiymatlari topiladi. Topilgan barcha qiymatlarni taqqoslanadi. Ularning eng kattasi funksiyaning *G* sohadagi eng katta qiymati, eng kichigi esa funksiyaning *G* sohadagi eng kichik qiymati bo'ladi.

Ba'zi hollarda funksiya chegaralangan yopiq sohaning chegarasidagi eng katta va eng kichik qiymatlarini topish uchun soha chegarasini ma'lum tenglamalarga ega qismlarga ajratgan ma'qul.

4-misol. $z=x^2-y^2$ funsiyaning $x^2+y^2\leq 9$ doiradagi eng katta va eng kichik qiymatlari topilsin.

Yechish. $z'_x=2x,\ z'_y=-2y$ xususiy hosilalarni nolga tenglashtirib

$$\begin{cases} 2x=0, \\ -2y=0 \end{cases}$$

sistemani yechsak $P_o(0;0)$ kritik nuqtaga ega bo'lamiz. $f(P_o)=z(0;0)=0$.

Endi doiraning chegarasi, ya'ni $x^2+y^2=3^2$ aylanada $z=x^2-y^2$ funksiyaning eng katta va eng kichik qiymatlarini topamiz. Aylanada x va y o'zgaruvchilar $y^2=9-x^2$ tenglik orqali bog'langanligi sababli aylana nuqtalari uchun $z=x^2-y^2$ funksiyani $z=x^2-(9-x^2)=2x^2-9$ ko'rinishdagi bir o'zgaruvchining funksiyasi shaklida tasvirlash mumkin, bunda $-3\leq x\leq 3$. Shunday qilib $z=x^2-y^2$ funksiyaning $x^2+y^2=9$ aylanadagi eng katta va eng kichik qiymatlarini topish bir o'zgaruvchining funksiyasi $z=2x^2-9$ ning $[-3;3]$ kesmadagi eng katta va eng kichik qiymatlarini topishga keltirildi. Bu funksiyaning

(-3;3) intervaldagi kritik nuqtalarini topamiz va funksiyaning kesmaning chetlaridagi qiymatlarini hisoblaymiz. $z'=(2x^2-9)'=4x$, $4x=0$. Bundan $x=0$ kritik nuqtaga ega bo'lamiz. $z(o)=-9$, $z(-3)=2\cdot(-3)^2-9=9$, $z(3)=2\cdot3^2-9=9$.

Bu qiymatlarni taqqoslab $z=x^2-y^2$ funksiyaning $x^2+y^2=9$ aylanadagi eng katta qiymatini *9* ga va eng kichik qiymarti *-9* ga teng ekanligiga ishonch hosil qilamiz.

Demak, $z=x^2-y^2$ funksiya $x^2+y^2\leq9$ doirada o'zining eng katta qiymati *9* ni $x^2+y^2=9$ aylananing $M_1(-3;0)$, $M_2(3;0)$ nuqtalarida va eng kichik *-9* qiymatini shu aylananing $M_1(0;3)$, $M_1(0;-3)$ nuqtalarida qabul qiladi.

O'z-o'zini tekshirish uchun savollar

1. Bir necha o'zgaruvchi funksiyasining maksimumi nima?
2. Bir necha o'zgaruvchi funksiyasining minimumi nima?
3. Funksiyaning maksimum nuqtasi nima?
4. Funksiyaning minimum nuqtasi nima?
5. Funksiyaning ekstremumi nima?
6. Ikki o'zgaruvchi funksiyasi uchun ekstremum mavjudligining zaruriy shartini ayting.
7. Funksiyaning kritik nuqtasi nima?
8. Ikki o'zgaruvchi funksiyasi uchun ekstremum mavjudligining yetarlilik shartini ayting.
9. Ikki o'zgaruvchi funksiyasining ekstremumini izlash qanday sxemaga asoslangan?
10. Ikki o'zgaruvchi funksiyasining yopiq chegaralangan sohadagi eng katta va eng kichik qiymatlari qanday topiladi?

Mustaqil yechish uchun mashqlar

Quyidagi funksiyalarning kritik nuqtalari topilsin.

1. $z=2x^2+xy^2+5x^2+y^2-4$. Javob: $(0;0)$, $\left(-\frac{5}{3};0\right)$, $(-1;2)$, $(-1;-2)$.

2. $z=e^{2x}(x+y^2+2y)$. Javob: $\left(\frac{1}{2},-1\right)$.

3. $z=2xy-3x^2-2z^2+10$ funksiyaning estremum nuqtasi topilsin. Javobi: $(0;0)$

4. $z=4(x-y)-x^2-y^2$ funksiyaning ekstremum nuqtasi topilsin Javob: $(2;-2)$.

5. $z=x^2+2xy-4x+8y+2$ funksiyaning $x=0, y=0, x=1, y=2$ to‘g‘ri chiziqlar bilan chegaralangan to‘g‘ri to‘rtburchakdagi eng katta va eng kichik qiymatlari topilsin. Javob: $(1;2)$ nuqtuda eng katta $z=19$ ga va $(1;0)$ nuqtada eng kichik $z=-1$ qiymatga ega. $(-4;6)$ kritik nuqta berilgan sohaga tegishli emas.

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35. TRIGONOMETRIK FUNKSIYALAR QATNASHGAN IFODALARNI INTEGRALLASH

36. BA‘ZI-BIR IRRATSIONAL FUNKSIYALARNI INTEGRALLASH

37. ANIQ INTEGRAL TUSHUNCHASI

38. ANIQ INTEGRALNING ASOSIY XOSSALARI. ANIQ INTEGRALNI HISOBLASH

39. XOSMAS INTEGRALLAR

40. ANIQ INTEGRALLARNI TAQRIBIY HISOBLASH

41. ANIQ INTEGRAL YORDAMIDA YUZNI HISOBLASH

42. EGRI CHIZIQ YOYINING UZUNLIGI

48. BIR NECHA O'ZGARUVCHI FUNKSIYALARINING XUSUSIY HOSILALARI VA DIFFERENSIALLANUVCHILIGI

49. BIR NECHA O'ZGARUVCHI FUNKSIYASINING TO'LA DIFFERENSIALI VA UNING TATBIQI

50. MURAKKAB FUNKSIYANING HOSILASI VA TO'LA DIFFERENSIALI

51. OSHKORMAS FUNKSIYANING HOSILASI. SIRTGA URINMA VA NORMAL

52. YUQORI TARTIBLI XUSUSIY HOSILALAR VA DIFFERENSIALLAR

53. IKKI O'ZGARUVCHI FUNKSIYASINING EKSTREMUMLARI

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100129, Toshkent sh., Navoiy ko'chasi, 30.